Sheaves as étale spaces and the inverse image sheaf

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① Sheaves as étale spaces (II.4, II.5 to II.6, MM)

② Inverse image sheaf (II.9, MM)

· X space ⇝ $\widehat{\mathcal{O}(X)} := \mathrm{Set}^{\mathcal{O}(X)^{op}}$

$\mathrm{Psh}(X) = [\mathcal{O}(X)^{op}, \mathrm{Set}]$

"presheaf category of presheaves over X."

① SHEAVES AS ÉTALE SPACES

· The moral of the story here is that sheaves over some topological space can be thought of as being the same as a certain type of bundle over that space.

· Let's do some review and setup first.

· First: what is it we mean when we say "bundle"?

def: Let $X \in \mathrm{Top}$. Objects of the slice category Top/X are called BUNDLES OVER X, sometimes denoted Bund X.

(ie) continuous maps $p: Y \to X$ and $f: p \to p' \in \mathrm{Top}/X$ is such that:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Y' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

Let $p: Y \to X$ be a bundle over X.

· A CROSS-SECTION of a BUNDLE p is a continuous map $s: X \to Y$ s.t. $ps = 1$

$$\begin{tikzcd} Y \arrow[r, "p"] & X \arrow[l, bend left, dashed, "s"] \arrow[loop right, "1_x"] \end{tikzcd}$$

(i.e.) cross-section = an arrow from $1_x: X \to X$ to $p: Y \to X$ in the slice category Top/X.

· We can also define what it means to take the cross-section over a single open set. In fact, the bundle restricts to each open set.

· If $U \in \mathcal{O}(X)$, consider $p|_U =: p_U : p^{-1}U \to U$ (a bundle over U).

Then, we have the following pullback diagram in Top:

$$\begin{tikzcd} p^{-1}U \arrow[r, "i"] \arrow[d, "p_U"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "p"] \\ U \arrow[r, "i"'] \arrow[ur, dashed, "s"] & X \end{tikzcd}$$

i : corresponding inclusion maps.

- A cross-section of the bundle p_U is called a CROSS-SECTION OF THE BUNDLE p OVER U, ⓔ $s: U \to Y$ s.t. $ps = i$

$$\begin{tikzcd} & Y \arrow[dr, "p"] & \\ U \arrow[ur, "s"] \arrow[rr, "i"'] & & X \end{tikzcd}$$

- Ultimately in this talk one of the big punchlines is the fact that sheaves over X can be regarded as a special type of bundle and vice versa. Let's start by describing how a bundle over X determines a sheaf over X.

def: Let $p: Y \to X$ be a bundle and $U \in \mathcal{O}(X)$. Define

$$\Gamma_p : \mathcal{O}(X)^{op} \to Set \qquad \text{by}$$

- $U \in \mathcal{O}(X) \rightsquigarrow \Gamma_p U := \{ s \; : \; s : U \to Y, \; ps = i : U \subseteq X \}$
- $V \subseteq U \in \mathcal{O}(X) \rightsquigarrow \Gamma_p U \to \Gamma_p V$ by restriction.

FACT: Γ_p is a sheaf.

HW: Check the conditions directly.

- Γ_p is called the SHEAF OF CROSS-SECTIONS of the BUNDLE p.

(sometimes use $\Gamma Y := \Gamma_p$)

- We have even more... for each bundle we have a sheaf and maps of bundles induce maps of sheaves by post-composition. So we actually have a functor:

def. Let $\Gamma : Bund\,X \to Sh(X)$ be defined by:

$$\Gamma : p \longmapsto \Gamma_p$$

$$f : p \to p' \in Bund\,X \longmapsto \Gamma_f : \Gamma_p \to \Gamma_q \quad \text{nat. trans. where}$$

$$\left(\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Y' \arrow[dl, "p'"] \\ & X & \end{tikzcd} \right)$$

$$(\Gamma_f)_U : \Gamma_p U \to \Gamma_q U \quad \text{for } U \in \mathcal{O}(X)$$

$$(s : U \to Y) \longmapsto (U \xrightarrow{s} Y \xrightarrow{f} Y')$$

Ex: Coverings $\tilde{X} \xrightarrow{p} X$: $\forall x \in X$, $\exists$ open nbhd $U \in \mathcal{O}(X)$, $x \in U$ s.t. $p^{-1}U = \coprod_i U_i$ for $U_i \in \mathcal{O}(X)$

and each U_i is mapped homeomorphically onto U by p.

"local homeomorphism in a strong sense"

- Many more examples of bundles — eg., principal G-bundles, discrete bundles

- Now, we can associate a bundle to each presheaf over X.
- To do that, we have to define the "germ" of a function.
- In complex analysis, two holomorphic functions are said to have the same "germ" at a point if their power series expansions centered around a are the same.
- For real-valued functions two functions have the same germ at a point x if they agree in some open neighbourhood of x. In this case, having the same germ at x means the functions take on the same value at x.
- We can define this property, which is actually an equivalence relation, for any presheaf over a topological space.

def: Let $P: \mathcal{O}(X)^{op} \to \mathbf{Set}$ be a presheaf, $x \in X$, $x \in U, V$ with $U, V \in \mathcal{O}(X)$, and $s \in PU$, $t \in PV$.

Then, $s \sim t$ if $\exists$ $W \subseteq U \cap V$, $W \in \mathcal{O}(X)$, $x \in W$ s.t.

"s and t have the same germ at x." $s|_W = t|_W \in PW$

- Fact: This is an equivalence relation.
- The equiv. class of any one such s is called the GERM of s at x, denoted $\text{germ}_x s$.
- $P_x := \{(U, s) : U \in \mathcal{O}(X), x \in U, s \in PU\} / \sim$ is called the STALK of P at x

$$= \{\text{germ}_x s : s \in PU, x \in U \in \mathcal{O}(X)\}$$

$$\rightsquigarrow P_x = \operatorname*{colim}_{x \in U} PU$$

∘ We are almost ready to define the functor, but we have to think about what we will do to morphisms of presheaves ⇝ so if I have a natural transformation, what map of bundles do I get?

- Let $h : P \to Q \in Psh(X)$ (so it is a natural transformation from the functor P to the functor Q).
- $x \in X$, $U \in \mathcal{O}(X)$ with $x \in U$.

Then we have the following diagram:

$$\begin{tikzcd}
PU \arrow[r, "h_U"] \arrow[d, "germ_x"'] & QU \arrow[d, "germ_x"] \\
P_x \arrow[r, dashed, "\exists!\, h_x"'] & Q_x
\end{tikzcd}$$

$$\begin{tikzcd}
s \in PU \arrow[r, mapsto] & h_U s \in QU \arrow[d, "germ_x"] \\
 & h_x(germ_x s) := germ_x(h_U s) \in Q_x
\end{tikzcd}$$

⇝ functor $Set^{\mathcal{O}(X)^{op}} \longrightarrow Set$

$$\begin{cases} P \longmapsto P_x \\ h \longmapsto h_x \end{cases}$$

"take the germ at x"

- And actually we can extend this idea to not just single stalks but to a bundle of stalks!
- So we're going to associate a particular bundle to each presheaf and we will do it functorially.

def: Let $\Lambda : Set^{O(X)^{op}} \longrightarrow Top/X$ be defined by:

$P \in Set^{O(X)^{op}} \rightsquigarrow \Lambda P := \Lambda_P = \coprod_{x \in X} P_x = \{ \text{all } germ_x s : x \in X,\ s \in PU \}$, sometimes called the "ÉTALE SPACE of (PRESHEAF) P"

$h : P \to Q \in Set^{O(x)^{op}} \rightsquigarrow \Lambda h := \coprod_{x \in X} (h_x : P_x \to Q_x)$

$\Lambda h : \Lambda_P \to \Lambda_Q$

- Equip Λ_P with the topology given by the basis:

$$B_{U,s} := \{ germ_x(s) : x \in U \} \qquad U \in \mathcal{O}(X),\ s \in PU$$

- sometimes called the ÉTALE TOPOLOGY

open set in Λ_P ⇝ a union of images of $\dot{s}$ as below.

Why is Λ_P a bundle?

Define $\pi_P : \Lambda_P \longrightarrow X$ by

$\mathrm{germ}_x s \longmapsto x$ This is just the canonical projection

- Let $s \in PU \rightsquigarrow \dot{s} : U \longrightarrow \Lambda_P$

$x \longmapsto \dot{s}x := \mathrm{germ}_x s \quad , x \in U$

By definition of π_P, $\dot{s}$ is a section of π_P.

$\rightsquigarrow \quad s \in PU \overset{\text{"replaced by"}}{\rightsquigarrow} \dot{s} : U \longrightarrow \Lambda_U$

- By the definition of the étale topology, then we have that
  - $\dot{s}$ is continuous $\forall U \in \mathcal{O}(X)$, open map and monic $\Rightarrow \dot{s} : U \to \dot{s}U$ is a homeomorphism.
  - π_P is continuous

$\therefore \Lambda_P, \pi_P$ is a bundle over X.

So for a presheaf P, we defined a bundle over X corresponding to P. We also notice that the projection π_P is a LOCAL HOMEOMORPHISM in the sense that each point of Λ_P has an open nbhd which is mapped by π_P homeomorphically onto an open subset of X. What do I mean?

Each $\mathrm{germ}_x s$ has open nbhd $\dot{s}U$

$\pi_P|_{\dot{s}U}$ inverse to $\dot{s} : U \to \dot{s}U$

$\Rightarrow \pi_P|_{\dot{s}U}$ is a homeomorphism to U.

So this bundle has a special property that locally the larger space is a mirror image of the base space. This type of bundle is the one corresponding to sheaves over X, and are called étale bundles or spaces.

def. A bundle $p: E \to X$ is called ÉTALE when it is a "local homeomorphism" in the following way:

$\forall e \in E$, $\exists$ $V \in \mathcal{O}(E)$ with $e \in V \subseteq E$ s.t. $pV \in \mathcal{O}(X)$ and

$p|_V : V \to pV$ is a homeomorphism.

eg) coverings

Remark: If $p: E \to X$ is étale, $U \in \mathcal{O}(X)$

$\rightsquigarrow$

$$\begin{tikzcd} E_U \arrow[r, "i"] \arrow[d, "p_U"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ U \arrow[r] & X \end{tikzcd}$$

$E_U \to U$ is étale over U

"A section of E will always mean a section s of E_U for some open U ($s: U \to E$ s.t. $ps = U \subseteq X$)."

• Now we have all the necessary ingredients for the following theorem.

Theorem (2)

$$\begin{tikzcd} Top/X \arrow[r, bend left, "\Gamma"] \arrow[r, phantom, "\top"] & Set^{\mathcal{O}(X)^{op}} \arrow[l, bend left, "\Lambda"] \end{tikzcd}$$

$\Lambda \dashv \Gamma$

for Γ and Λ as previously mentioned, i.e.:

- $\Gamma : f \mapsto \Gamma_f$ (sheaf of cross-sections of $f: Y \to X$)
- $\Lambda : P \mapsto \Lambda_P$ (étale space of a presheaf P)

There are nat. transformations

- $\eta_P : P \to \Gamma \Lambda P$ (unit)

and

- $\varepsilon_f : \Lambda \Gamma f \to f$ (counit)

such that $\Lambda \dashv \Gamma$.

$\boxed{\text{Proof}}$ (Idea)

- Γ, Λ are functors
- $\eta_U : (s \in PU) \mapsto \left(\begin{array}{l} \dot s : U \to \Lambda_P \\ \quad x \mapsto \text{germ}_x s \end{array} \right)$

  $U \in \mathcal{O}(X)$

  Since the restriction operator commutes with η, we have that η is a nat. trans.

- $\varepsilon_f : \underbrace{\Lambda}\Gamma f \to f$

  every point is of the form $\dot s x (= \text{germ}_x s)$ for some $x \in X$ and some cross-section $s : U \to Y$.

  $\overset{def}{\rightsquigarrow} \quad \varepsilon_f \dot s x := sx \in Y, \quad x \in U, \; s \in \Gamma_f U$

  <u>To check:</u>

  - independence on rep of $\text{germ}_x s$
  - ε_f is continuous – ie, a map of bundles
  - natural in Y

- The adjunction: check it! (HW).

Verify that

$$\begin{tikzcd} \Gamma \arrow[r, "\eta\Gamma"] \arrow[rr, bend left, "1"] & \Gamma\Lambda\Gamma \arrow[r, "\Gamma\varepsilon"] & \Gamma \end{tikzcd}$$

and

$$\begin{tikzcd} \Lambda \arrow[r, "\Lambda\eta"'] \arrow[rr, bend left, "1"] & \Lambda\Gamma\Lambda \arrow[r, "\varepsilon\Lambda"'] & \Lambda \end{tikzcd}$$

(ie) Let $f : Y \to X$ be a bundle, $s \in \Gamma_f U$ a cross-section. Then,

$$\begin{tikzcd} \Gamma \arrow[r, "\eta\Gamma"] \arrow[rr, bend left, "1"] & \Gamma\Lambda\Gamma \arrow[r, "\Gamma\varepsilon"] & \Gamma \\ s \arrow[r, mapsto] & \overset{1}{\dot s} \arrow[r, mapsto] & s \end{tikzcd}$$

$$\begin{tikzcd} \Lambda \arrow[r, "\Lambda\eta"'] \arrow[rr, bend left, "1"] & \Lambda\Gamma\Lambda \arrow[r, "\varepsilon\Lambda"'] & \Lambda \\ \text{germ}_x s \arrow[r, mapsto] & \text{germ}_x \dot s \arrow[r, mapsto] & \dot s x = \text{germ}_x s \end{tikzcd}$$

$\boxed{\text{Prop}}$ If P is a sheaf, then η_P is an isomorphism.

$\boxed{\text{proof}}$ Check that it is injective and surjective.

$\rightsquigarrow$ Use definitions. $\square$

$\boxed{\text{Prop}}$ If $f: Y \to X$ is étale ($\in \mathrm{Etale}\,X \subseteq \mathrm{Bund}\,X$), then ε_f is an iso.

↖ full subcat consisting of étale bundles.

$\boxed{\text{Proof}}$

Inverse:

$$\theta_f : f \longrightarrow \Lambda\Gamma f$$

where $\theta_f\, y := \dot{s}x$ ($=\mathrm{germ}_x s$) $\in \Lambda\Gamma f$

for $x \in X$ with $fy = x$, $s: U \to Y$. $sx = y$
$\underline{\underline{x \in U}}$

This def is independent on the choice of cross-section s (as f is a local homeomorphism (By definition of étale $\overset{\text{PROP}}{\Longrightarrow}$ f and all its sections are $\underline{\underline{\text{open}}}$ maps. If s and t are 2 sections, the set $W = \{x : sx = tx\}$ is open in X)

- θ_f is continuous
- θ_f is the inverse of ε_f by $\underline{\underline{\text{def}}}$ ($\varepsilon_f\, \dot{s}x := sx \in Y$, $x \in U$, $s \in \Gamma_f U$).

$\square$

$\boxed{\text{Cor}}$ Γ and Λ restrict to an equiv of categories:

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[r, shift left, "\Lambda"] & \mathrm{Etale}\,X \arrow[l, shift left, "\Gamma"] \end{tikzcd}$$

- There's also a more general result that I won't state here — (ie) when an adjunction restricts to an equivalence of subcategories, and those subcategories are also "special" in that they are $\underline{\underline{\text{reflective}}}$ and $\underline{\underline{\text{coreflective}}}$.

inclusion functor has a left adjoint.

$\mathrm{Sh}(X) \subset O(X)$ is REFLECTIVE.

- So what does this adjunction give us?

$\boxed{\text{Cor}}$ Given sheaves $F, G \in Sh(X)$, a morphism $h: F \to G$ of sheaves may be described in any one of the following 3 equivalent ways; as a:

(i) nat. trans $F \overset{h}{\Rightarrow} G$

(ii) cont. map $\Lambda_h : \Lambda F \to \Lambda G$ of bundles over X

(iii) family $\underbrace{\{h_x : F_x \to G_x\}}_{\text{fiber over each } x \in X}$ s.t. $\forall$ U open and $s \in FU$, the function $x \longmapsto h_x(s_x)$ is continuous $U \to \Lambda G$.

- This leads to a very useful result.

$\boxed{\text{Prop}}$ A map $h: F \to G \in Sh(X)$ is epic (resp. monic) iff $\forall x \in X$, $h_x : F_x \to G_x$ (the map of stalks) is surjective (resp. injective) as a set map.

- The proof in the book uses an adjunction between the stalk functor and the so-called skyscraper functor.

$Sky_x : Set \to Sh(X)$

$$A \longmapsto Sky_x(A) : U \mapsto \begin{cases} A & \text{if } x \in U \\ 1 & \text{else} \end{cases}$$

some fixed 1-element set

$stalk_x : Sh(X) \to Set$

$$F \longmapsto F_x = \operatorname*{colim}_{x \in U} FU$$

Lemma:

$\Rightarrow$ For each $x \in X$, $Stalk_x \dashv Sky_x$

§2. Inverse image sheaf (II.9)

def. Let $f: X \to Y$ be a continuous map of spaces. Let F be a sheaf over X.
The direct image of F under f is the induced sheaf on Y:

$$(f_* F)(V) := F(f^{-1}V) \quad \text{for each } V \in \mathcal{O}(Y)$$

$$Ff^{-1}: \mathcal{O}(Y)^{op} \to \text{Set}$$

- note that for $U \subseteq V$ in $\mathcal{O}(Y)$ and $s \in f_* F V$, we have that

$$f_* F(U \subseteq V)s = F(f^{-1}U \subseteq f^{-1}V)s$$

$$\rightsquigarrow \quad f_*: \text{Sh}(X) \to \text{Sh}(Y) \text{ is a functor!}$$

The question: What about the other direction?

Rmk Let $p: E \to Y$ be a bundle and $f: X \to Y$ continuous map of spaces. Then, the pullback of p along f, f^*, is the usual pullback in Top. Note that the pullback yields a bundle

$$f^* E \to X$$

and $f^*: \text{Bund}\,Y \to \text{Bund}\,X$ is a functor ← Prop 4.5 in YS

"étale" is preserved under pullbacks

Lemma The pullback of an étale bundle along a continuous map is also étale.

proof (we have) The following pullback diagram: "fiber product"

$$\begin{tikzcd}
f^* E = X \times_Y E \arrow[r] \arrow[d, "f^* p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \ni e \arrow[d, "p"] \\
X \arrow[r, "f"] & Y \\
\overset{\rotatebox{90}{$\in$}}{x} &
\end{tikzcd}$$

$$\begin{tikzcd}
D \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right, "\forall"'] & & \\
& \hat{A} \arrow[r, "f^* g"] \arrow[d, "g^* f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\
& C \arrow[r, "g"'] & A
\end{tikzcd}$$

from the first seminar (Jon's talk)

$$\left\{ \begin{tikzcd} D \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & A \end{tikzcd} \text{ in } \mathcal{C} \right\} \simeq \mathcal{C}(D, C \times_A B)$$

"pullback data"

Now, consider any point in the pullback, $\langle x, e\rangle \in f^* E$.

(ie) $x \in X$ and $e \in E$ where $fx = pe$.

As p is étale, $\exists$ nbhd U of e s.t. $p|_U$ is a homeomorphism.

Then, define ↙ pre-image of open!

$$V := f^{-1}(pU) \times U$$

$\Rightarrow V$ is an open nbhd of $\langle x, e\rangle$ in the product $X \times E$.

$\Rightarrow V \cap (X \times_Y E)$ is open in induced topology on $X \times_Y E$ and is mapped homeomorphically onto $f^{-1}(pU) \subseteq X$.

$\Rightarrow f^* p$ is a local homeomorphism. $\square$

$\therefore$ Each continuous $f: X \to Y$ gives a functor $Sh(X) \longrightarrow Sh(Y)$ via equivalence of categories Etale X and $Sh(X)$.

Remark Consider the pullback functor f_0^* as before.
By the equivalence of categories $Sh(X) \rightleftarrows \text{Etale}\, X$, then we get :

$$\begin{tikzcd}
Sh(Y) \arrow[r, "\Lambda"] \arrow[rrr, bend right] & \text{Etale}\, Y \arrow[r, "f_0^*"] & \text{Etale}\, X \arrow[r, "\Gamma"] & Sh(X)
\end{tikzcd}$$

f^*

def. If G is a sheaf on Y, the value $f^*(G) \in Sh(X)$ is called the INVERSE IMAGE of G (under f)

useful for the inv image - direct image adjunction (next prop)

some more remarks By def^n, it is determined by its étale bundle by the pullback :

Let $t': U \to \Lambda G$.
By the pullback, $\exists!$ $U \to \Lambda(f^*G) \in Top/X$
Apply the equivalence!

$$\Gamma U \longrightarrow \Gamma\Lambda f^*G \in Sh(X)$$

Yoneda embedding: $y_U := \mathcal{O}(X)(-, U)$, $y_U \Rightarrow f^*G$
$\Rightarrow \exists\ t \in f^*G(U)$.

$$\begin{tikzcd}
\Lambda f^*G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda G \arrow[d, "p"] \\
X \supseteq U \arrow[u, bend left, "t_s"] \arrow[ur, "t'"] \arrow[r] & Y
\end{tikzcd} \qquad (*)$$

$$\xrightarrow[f|_U]{f}$$

For any open $U \subseteq X$, a section $t \in (f^*G)(U)$ of the INVERSE IMAGE SHEAF can thus be described (by def^n of a pullback) as a continuous map

$$t': U \to \Lambda G \quad \text{s.t.}$$

$$* \ pt' = f|_U : U \to Y$$

AKA... a section t corresponds to a LIFTING t' to ΛG of the map $f: U \to Y$, (ie)

$$\begin{tikzcd}
& \Lambda G \arrow[d, "p"] \\
U \arrow[ur, dashed, "t'"] \arrow[r, "f|_U"'] & Y
\end{tikzcd} \qquad (**)$$

$$t \overset{\text{corresponds to}}{\leftrightsquigarrow} t'$$

In PARTICULAR... If $V \subseteq Y$ is OPEN and $s \in G(V)$, obtain a SECTION

$$\dot{s} : V \to \Lambda G \qquad (\dot{s}_y := \text{germ}_y s \ , y \in V)$$

Hence, by COMPOSITION, get a map t' as in (**), therefore by the pullback (*) a SECTION

$$(***) \quad t_s : f^{-1}(V) \to \Lambda(f^*G)$$

Specifically, by the pullback (*), a point of $\Lambda(f^*G)$ is of the form:

$$(x, \text{germ}_{f(x)} s) \quad \text{where} \quad x \in X \text{ and } \text{germ}_{f(x)}(s) \in (\Lambda G)_{f(x)}$$

the germ of some $s \in GV$ for some open nbhd $V \subseteq Y$ of $f(x)$.

Writing points of $\Lambda(f^*G)$ as such pairs, we can write t_s as in (***) explicitly as:

$$(****) \qquad t_s(x) = (x, \text{germ}_{f(x)}(s)) \qquad \text{for } x \in f^{-1}(V)$$

Since $\Lambda f^*G \to X$ is étale, the image of each section $t_s : f^{-1}(V) \to \Lambda(f^*G)$ is OPEN. Also, for all open $V \subset Y$ and all $s \in GV$, the images of each section cover Λf^*G. This holds because of (****) since every point of Λf^*G has the form $(x, \text{germ}_{f(x)} s)$ for some x and s.

$\Rightarrow$ If T is a topological space
$K : \Lambda f^*G \to T$ function

Then,

$$(K \text{ is continuous}) \Leftrightarrow \left(\begin{array}{l} K \circ t_s : f^{-1}(V) \to T \text{ is continuous} \\ \text{for every } V \text{ and } s \end{array} \right)$$

ⓘⓔ "continuous = continuous on each section"

used in the proof of the following thm.

Inverse image - direct image adjunction.

$\boxed{\text{Thm}}$ $f : X \to Y$ continuous.

The inverse image functor f^* is left adjoint to the direct image functor f_* :

$$\begin{tikzcd} Sh(X) \arrow[r, bend right, "f_*"'] \arrow[r, phantom, "\perp"] & Sh(Y) \arrow[l, bend right, "f^*"'] \end{tikzcd} \qquad f^* \dashv f_*$$

← left.

$\boxed{\text{proof}}$ ← quite long (> 1 page of text), so maybe will give a pipeline of the ideas used in the proof instead.

Let $F \in Sh(X)$ and $G \in Sh(Y)$

GOAL: $Sh(X)(f^*G, F) \cong Sh(Y)(G, f_* F)$ (natural in both F and G)

* Reminiscent of the sheaf morphism equivalent defs in corollary (6.5) :

$$\begin{aligned} Sh(X)(f^*G, F) &\cong Et_X(\Lambda f^* G, \Lambda F) && (1) \\ &\cong K(\Lambda f^* G, \Lambda F) && (2) \\ &\cong Sh(Y)(G, f_* \Gamma \Lambda F) && (3) \\ &\cong Sh(Y)(G, f_* F) && (4) \end{aligned}$$

$\boxed{(1)}$

- $Et_X(\Lambda f^* G, \Lambda F)$: "set of maps of étale bundles over X"

(1) holds by the categorical equiv. between sheaves and étale bundles.

Side note: The constructions will make it "clear" that the isos are natural in F and G.

$\boxed{(4)}$ $Sh(Y)(G, f_* \Gamma \Lambda F) \cong Sh(Y)(G, f_* F)$

This holds by the fact that F is a sheaf ⇒ The unit $\eta : F \to \Gamma \Lambda F$ is an isomorphism.

Then, compose with the inverse of $f_* \eta : f_* F \xrightarrow{\sim} f_* \Gamma \Lambda F$

$\boxed{(2)}$ $Et_X(\Lambda f^* G, \Lambda F) \cong K(\Lambda f^* G, \Lambda F)$: K is yet to be defined!

def: $K(\Lambda f^* G, \Lambda F) := \{ k : \Lambda f^* G \to \Lambda F \mid \forall V \in \mathcal{O}(Y), s \in G(V), k \circ t_s : f^{-1}(V) \to \Lambda f^* G \to \Lambda F \}$

is CONTINUOUS

from above remarks

ⓘⓔ "FUNCTIONS CONTINUOUS ON SECTIONS"

By the preceding remarks (continuous = continuous on each section), then $\boxed{(2)}$ holds:

$$Et_X(\Lambda f^* G, \Lambda F) \cong K(\Lambda f^* G, \Lambda F)$$

$\boxed{(3)}$ $K(\Lambda f^*G, \Lambda F) \cong Sh(Y)(G\ f_*\Gamma\Lambda F)$

[method: by constructing maps + showing they are mutually inverse.]

left to right: $K(\Lambda f^*G, \Lambda F) \to Sh(Y)(G, f_*\Gamma\Lambda F)$

·Given $K: \Lambda f^*G \to \Lambda F \in K(\Lambda f^*G, \Lambda F)$, define

$\tau_K : G \to f_*\Gamma\Lambda F$ by:

by defⁿ of the direct image.

For $V \in \mathcal{O}(Y)$ $\rightsquigarrow$ $(\tau_K)_V : G(V) \to f_*(\Gamma\Lambda|$

is s.t. $s \longmapsto (\tau_K)_V(s)$

for t_s as in the remarks. (1e)

$$f^{-1}(V) \xrightarrow{t_s} \Lambda G \quad K$$

$(\tau_K)_V(s)$

As K is continuous, by the "$\Leftarrow$" statement in the preceding rema

continuous $\forall$ V, s.

$\Rightarrow$ Each $(\tau_K)_V(s)$ is continuous.

right to left: $Sh(Y)(G, f_*\Gamma\Lambda F) \to K(\Lambda f^*G, \Lambda F)$

Let $\tau: G \to f_*\Gamma\Lambda F$ be a natural transformation.

Recall from the preceding remarks that a point of Λf^*G has the general form $(x, germ_{f(x)}(s))$ for $s \in GV$ for some $V \in \mathcal{O}(Y)$ s.t. $f(x) \in V$. (1e) in $(****)$

Define a function

$k_\tau : \Lambda f^*G \to \Lambda F$ as follows:

$$k_\tau(x, germ_{f(x)}(s)) := \tau_V(s)(x)$$

- It is well-defined: does not depend on $s \in GV$ chosen to represent the germ $germ_{f(x)}(s)$

RECALL:

$$K(\Lambda f^* G, \Lambda F) := \{ k : \Lambda f^* G \to \Lambda F \mid \forall V \in \mathcal{O}(Y), s \in G(V), \underbrace{k \circ t_s : f^{-1}(V) \to \Lambda f^* G \to \Lambda F}_{\text{IS CONTINUOUS}} \}$$

Want to show: $k_\tau \in K(\Lambda f^* G, \Lambda F)$

To this end...

- If $V \in \mathcal{O}(Y)$ (open in Y), $s \in GV$, then by the preceding remarks, specifically, by:

$$(****) \qquad t_s(x) = (x, \text{germ}_{f(x)}(s)) \qquad \text{for } x \in f^{-1}(V)$$

Then,

$$\begin{aligned} (k_\tau \circ t_s)(x) &= k_\tau(x, \text{germ}_{f(x)}(s)) \\ &= \tau_V(s)(x) \qquad (*) \end{aligned}$$

$\rightsquigarrow$ $k_\tau \circ t_s = \tau_V(s)$ is a continuous map $f^{-1}(V) \longrightarrow \Lambda F$.

$\Rightarrow$ $k_\tau \in K(\Lambda f^* G, \Lambda F)$.

They are MUTUALLY INVERSE :

$$K(\Lambda f^* G, \Lambda F) \rightleftarrows Sh(Y)(G, f_*(\Gamma \Lambda F))$$

$$k \longmapsto \tau_k$$

$$k_\tau \longleftarrow \tau$$

Let $\tau : G \to f_*(\Gamma \Lambda F)$ be a natural transformation.

Then, for any open $V \in \mathcal{O}(Y)$ and any $s \in GV$,

BY DEFINITION: $\tau_{(k_\tau)}(s) = k_\tau \circ s$

$= \tau_V(s)$ by $(*)$

On the other hand, suppose $k : \Lambda f^* G \longrightarrow \Lambda F \in K$. For any point $(x, \text{germ}_{f(x)}(s)) \in \Lambda f^* G$,

$k_{(\tau_k)}(x, \text{germ}_{f(x)}(s)) = (\tau_k)_V(s)(x)$ by definition of k_τ.

$= (k \circ t_s)(x)$ by definition of τ_k

$$= K(t_s(x))$$
$$= K(x, \mathrm{germ}_{f(x)}(s)) \quad \text{by} \quad (****) \text{ again}$$

$\leadsto$ The maps $K \mapsto T_K$ and $T \mapsto K_T$ are mutually inverse and so the iso $\boxed{3}$ holds.

$\square$

Proposition 4.16 3 in MM

Let $f: X \to Y$ be a continuous map of spaces. The inverse image sheaf functor $f^*: Sh(Y) \to Sh(X)$ preserves all finite limits.

The proof recipe

① Use the equivalence $Sh(X) \leftrightarrow \text{Etale } X$ to reduce the problem (land in the slice cat)

② The restriction of the pullback in the slice categories preserves all finite limits (it has a left adjoint)

③ $\text{Etale } X \subseteq \text{Bund } X := \text{Top}/X$ is closed under finite limits : has fin. lim s iff
- bin. products
- terminal obj.
- equalizers

proof

By the equivalence of categories between étale bundles and sheaves over a space X, then the inverse image sheaf functor $f^*: Sh(Y) \to Sh(X)$ is a pullback.

It suffices to show that $f^*: \text{Etale } Y \to \text{Etale } X$ preserves finite limits.

by the equiv of cats.

Also note that f^* is the restriction of the pullback in the slice categories $\text{Top}/X =: \text{Bund } X$ and $\text{Top}/Y =: \text{Bund } Y$:

$$\begin{tikzcd}
\text{Top}/Y \arrow[r, "f^*"] & \text{Top}/X \\
\text{Etale } Y \arrow[u, "i_Y"] \arrow[r, "f^*"'] & \text{Etale } X \arrow[u, "i_X"']
\end{tikzcd}$$

⊛ ?

Claim: $f^*: \text{Top}/Y \to \text{Top}/X$ has a left adjoint (Σ_f = "compose with f") and so preserves all finite limits.

Remains to show: $\text{Etale } X \subseteq \text{Top}/X$ is closed under finite limits. (the inclusion functor preserves finite limits)

Show this by the equivalent characterization:

C has finite limits $\overset{\text{char}}{\Leftrightarrow}$
1. binary products
2. equalizers
3. terminal object.

1. Binary products

Suppose both $E \to X$ and $E' \to X$ are étale over X.
Their pullback $E \times_X E' \in \mathrm{Top}$ is their product in Top/X.
By the lemma (at the beginning of this section), this pullback $E \times_X E'$ is étale in X,
hence it is also the product (of E and E') in $\mathrm{Etale}\,X$.
$\Rightarrow$ $\mathrm{Etale}\,X$ is closed under binary products.

2. Equalizers

If $f, g : E \to E' \in \mathrm{Etale}\,X$ are maps of bundles $\Rightarrow$ Equalizer in Top is also étale in X.

An exercise?

$$\begin{tikzcd}
eq \arrow[r] & E \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr, bend right] & & E' \arrow[dl, bend left] \\
K \arrow[u, dashed, "\exists!"] \arrow[ur, "\forall"'] & & X &
\end{tikzcd}$$

The equalizer of $f, g = \{ e \in E \mid f(e) = g(e) \} \subseteq E$

3. Terminal object

What's terminal in Top/X? $id_X : X \to X$! (It's a slice category!)
And clearly $id_X \in \mathrm{Etale}X$.

$\rightsquigarrow$ $\mathrm{Etale}\,X \subseteq \mathrm{Top}/X$ is closed under finite limits $\Rightarrow$ f^* preserves finite limits!

$\square$