Lattice and Heyting algebra objects in a topos

§ IV.8

Recall: • A lattice is a poset that has all binary products and coproducts (when viewed as a category)

ⓘⓔ a set with 2 distinguished elements $\bot$, $\top$ and two associative and commutative binary operations $\wedge$ and $\vee$ s.t.:

meet join

$$x \wedge x = x \qquad x \vee x = x \qquad \text{idempotent}$$

$$\top \wedge x = x \qquad \bot \vee x = x$$

$$x \wedge (y \vee x) = x = (x \wedge y) \vee x \qquad \text{absorption}$$

• A Heyting algebra is a Cartesian closed poset with all finite products and coproducts.

ⓘⓔ a lattice with $\bot$ and $\top$ such that for any pair of elements x, y, there is an exponential $y^x =: (x \Rightarrow y)$ characterized by (the adjunction)

$$z \leq (x \Rightarrow y) \quad \text{iff} \quad z \wedge x \leq y$$

Fact: The implication $\Rightarrow$ in a Heyting algebra satisfies:

(1) $(x \Rightarrow x) = \top$

(2) $x \wedge (x \Rightarrow y) = x \wedge y$

(3) $y \wedge (x \Rightarrow y) = y$

(4) $x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z)$

The "equational identities of a Heyting alg."

Now we will head in a different direction and think about lattices and Heyting algs that exist within a category other than Set.

• C: category with finite limits

def. A LATTICE OBJECT / INTERNAL LATTICE in C is $L \in Ob(C)$ together with 2 arrows:

"meet" $\quad \wedge : L \times L \to L$

"join" $\quad \vee : L \times L \to L$

s.t. the identities in the equational def^n of a lattice create a commutative diagram.

ⓔⓖ idempotent: $x \wedge x = x$

$$\begin{tikzcd}
(x,x) \arrow[rrr, mapsto] & L \times L \arrow[r, "\wedge"] & L & x \wedge x \\
x \arrow[u, mapsto] & L \arrow[u, "\delta"] \arrow[ur, "1"'] & &
\end{tikzcd}$$

eg. The absorption law $x \wedge (y \vee x) = x = (x \wedge y) \vee x$ as a commutative diagram:

$$\begin{tikzcd}
L & & L \times L \arrow[ll, "\wedge"'] \\
L \times L \arrow[u, "p"] \arrow[d, "p"'] \arrow[r, "\delta \times 1"'] & L \times L \times L \arrow[r, "1 \times \tau"'] & L \times L \times L \arrow[u, "1 \times \vee"'] \arrow[d, "\wedge \times 1"] \\
L & & L \times L \arrow[ll, "\vee"]
\end{tikzcd}$$

LHS: $(x,y) \mapsto (x,x,y) \mapsto (x,y,x) \mapsto (x, y \vee x) \mapsto x \wedge (y \vee x)$; $(x,y) \mapsto x$

RHS: $(x,y) \mapsto (x,x,y) \mapsto (x,y,x) \mapsto (x \wedge y, x) \mapsto (x \wedge y) \vee x$; $(x,y) \mapsto x$

$p := \pi_1$ (proj onto first factor)

$\tau : L \times L \to L \times L$ (twist map that interchanges the factors of the product)

$\delta : L \to L \times L$: diagonal map

• such a lattice object has $\top, \perp$ when there are arrows (where $1 \in C$ is terminal)

"top" $\quad \top : 1 \to L$

"bottom" $\quad \perp : 1 \to L$

such that:

Really: $\perp(1)$ (the bottom arrow applied to 1)

$x \vee \perp = x$

and

$x \wedge \top = x$

ie They make the following composites the identity:

$$\begin{tikzcd}
L \arrow[r, "\cong"] \arrow[rrr, bend right, "1"'] & L \times 1 \arrow[r, "1 \times \perp"] & L \times L \arrow[r, "\vee"] & L
\end{tikzcd}$$

$(x,1) \mapsto (x, \perp(1)) \mapsto x \vee \perp(1)$

$$\begin{tikzcd}
L \arrow[r, "\cong"] \arrow[rrr, bend right, "1"'] & L \times 1 \arrow[r, "1 \times \top"] & L \times L \arrow[r, "\wedge"] & L
\end{tikzcd}$$

<u>def.</u> L a lattice object.

L is an <u>INTERNAL HEYTING ALGEBRA IN C</u> (or just "Heyting algebra") if $\exists$ binary operation $\Rightarrow : L \times L \to L$ satisfying the diagrammatic version of the identities for $\Rightarrow$.

$\Rightarrow : (x,x) \mapsto (x \Rightarrow x)$

$\top(1) =$ "one" in the lattice

1 element = "TOP"

$\top : 1 \to L$

(1)

$$\begin{tikzcd}
1 \arrow[r, "\top"] & L \\
L \times L \arrow[u] \arrow[ur, "\Rightarrow"']
\end{tikzcd}$$

(x,x) $\quad$ $(x \Rightarrow x)$

$x \Rightarrow x = \top(1)$

(2) $x \wedge (x \Rightarrow y) = x \wedge y$

$$\begin{tikzcd}
L \times L \arrow[r, "\wedge"] \arrow[d, "\delta \times 1"'] & L \\
L \times L \times L \arrow[r, "1 \times \Rightarrow"] & L \times L \arrow[u, "\wedge"']
\end{tikzcd}$$

$(x,y) \mapsto x \wedge y$

$x \wedge (x \Rightarrow y)$

$(x,x,y) \mapsto (x, x \Rightarrow y)$

• For L a lattice object, can define corresponding partial order on L by:

$$x \leq y \quad \text{iff} \quad x \wedge y = x$$

<u>def</u>. Subobject $\leq_L$ of $L \times L$ is the equalizer:

$$\begin{tikzcd}
\leq_L \arrow[r, "e"] & L \times L \arrow[r, shift left, "\wedge"] \arrow[r, shift right, "p"'] & L \\
D \arrow[u, dashed, "\exists !"] \arrow[ru, "\forall"'] & &
\end{tikzcd}$$

$x \times y \longmapsto x$

$x \times y \longmapsto x \wedge y$

$\boxed{\text{FACT}}$ $(L, \leq_L)$ is an INTERNAL POSET • a poset that lives in the ambient cat/topos $\mathcal{C}$

$\leq_L$ reflexive:

$$\begin{tikzcd}
L \arrow[r, "\delta"] \arrow[rd] & L \times L \\
& \leq_L \arrow[u]
\end{tikzcd}$$

• Transitivity of $\leq_L$:

• The subobject $\langle \pi_1 e v, \pi_2 e u \rangle : C \rightarrowtail L \times L$ factors through $e : \leq_L \rightarrowtail L \times L$ where C is the following pullback, with projections u and v:

$$\begin{tikzcd}
C \arrow[rr, "u"] \arrow[dd, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & & \leq_L \arrow[d, tail, "e"] \\
& & L \times L \arrow[d, "\pi_1"] \\
\leq_L \arrow[r, tail, "e"'] & L \times L \arrow[r, "\pi_2"'] & L
\end{tikzcd}$$

??

(second def^n)

def. An <u>INTERNAL HEYTING ALGEBRA</u> of a topos $\mathcal{E}$ is an internal lattice of $\mathcal{E}$ with an additional binary operation $\Rightarrow : L\times L \to L$ such that the 2 subobjects P and Q of $L\times L\times L$ defined by the 2 pullback squares below are equivalent subobjects:

$$\begin{tikzcd}
P \arrow[r, dashed] \arrow[d, dashed] & \leq_L \arrow[d, tail, "e"'] & Q \arrow[l, dashed] \arrow[d, dashed] \\
L\times L\times L \arrow[r, "\wedge\times 1"'] & L\times L & L\times L\times L \arrow[l, "1\times\Rightarrow"]
\end{tikzcd}$$

ie $\overset{\text{RHS}}{z \leq (x\Rightarrow y)}$ iff $\overset{\text{LHS}}{z\wedge x \leq y}$ $(*)$

(equational def^n of Heyting alg.)

- This def^n is equivalent to the other one (equational identities for $\Rightarrow$)
  - equivalence proof: Yoneda applied to hom-sets $\mathrm{Hom}(X, L)$

def. A <u>HOMOMORPHISM OF LATTICES</u> (or of <u>HEYTING ALGEBRA OBJECTS</u>) $L \to L'$ is $f: L\to L'$ that commutes with all operations.

B

$$\begin{tikzcd}
1 \arrow[r, "\top"] \arrow[d, equal] & L \arrow[d, "f"'] & L\times L \arrow[l, "\wedge"'] \arrow[d, "f\times f"] \\
1 \arrow[r, "\top"'] & L' & L'\times L' \arrow[l, "\wedge"]
\end{tikzcd}$$

operation replacements

$\perp$ $\vee$, $\Rightarrow$

$\boxed{\text{Thm 1 (EXTERNAL)}}$

Let $A \in Ob(\mathcal{E})$.
The poset $Sub_{\mathcal{E}} A$ of subobjects of A has the structure of a Heyting algebra. This structure is natural in A.

↳ ie The pullback along any morphism $k: A \longrightarrow B$ induces a map k^{-1} of Heyting algebras as in:

$$\begin{tikzcd}
Sub_{\mathcal{E}} B \arrow[r, "k^{-1}"] \arrow[d, "i_B"'] & Sub_{\mathcal{E}} A \arrow[d, "i_A"] \\
\mathcal{E}/B \arrow[r, "k^*"'] & \mathcal{E}/A
\end{tikzcd}$$

slices! ⇝ $\mathcal{E}/B$

The "change-of-base" functor → k^*

$$\begin{tikzcd}
X' \arrow[dr, "f'"'] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& A \arrow[r, "k"] & B \\
& X &
\end{tikzcd}$$

(p.192) · Consider any $k: A \to B$ in a topos $\mathcal{E}$.
Pullback along k then turns each object $f: X \to B$ of $\mathcal{E}/B$ into an object f' of $\mathcal{E}/A$:

$A \times_B B$ (the pullback) $=: X'$

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
A \arrow[r, "k"'] & B
\end{tikzcd}$$

$k^*: f \mapsto f'$

⇝ defines a "change-of-base" functor / "PULLBACK FUNCTOR"
$k^*: \mathcal{E}/B \longrightarrow \mathcal{E}/A$

take an element of $\mathcal{E}/B$

Use map $k: A \to B$

$\boxed{\text{Proof of Thm 1}}$

- $A \in Ob(\mathcal{E}) \rightsquigarrow \mathrm{Sub}_{\mathcal{E}}(A)$ has lattice structure.
  ("factorization + images" section)

$\boxed{\text{PROP 3 in §6}}$ Sub A is a lattice.
For each $k: A \to B$, the pullback along k is a morphism $k^{-1}: \mathrm{Sub}B \to \mathrm{Sub}A$ of posets (ie, a functor). This functor has a left adjoint $\exists_k : \mathrm{Sub}A \to \mathrm{Sub}B$
$S \mapsto$ (image in B under k)

$(\exists_k \dashv k^{-1})$

- $S \rightarrowtail A$, $T \rightarrowtail A$: two subobjects in Sub A.
- Form their "intersection" (greatest lower bound in Sub$A$) by taking the pullback:

toposes have pullbacks.

$$\begin{tikzcd}
S \cap T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, tail] \\
S \arrow[r, tail] & A
\end{tikzcd}$$

- For their "union" (least upper bound), form the coproduct $(S+T)$ in the topos.
  Then, $\exists!$ $S+T \to A$ (by def$^{\underline{n}}$ of coproduct).
  This has an image M (by result on factorizations + images)
  $\rightsquigarrow M \rightarrowtail A$ is a subobject that contains both S and T

$$\begin{tikzcd}
S+T \arrow[dr, "epi"] \arrow[ddrr, bend right, red, "\exists!\ \text{coproduct}"'] & & T \arrow[ll] \arrow[dd, tail] \\
& M \arrow[dr, tail, "mono"] & \\
S \arrow[uu] \arrow[rr, tail] & & A
\end{tikzcd}$$

$M \rightarrowtail A \in \mathrm{Sub}\,A$

So we have $\wedge : \mathrm{Sub}A \times \mathrm{Sub}A \to \mathrm{Sub}A$ and it is natural in A
(The pullback functor $k^{-1}: \mathrm{Sub}B \to \mathrm{Sub}A$ is s.t. $k^{-1}(S \wedge T) = k^{-1}(S) \cap k^{-1}(T)$)
Using the iso:

$$\mathrm{Hom}(A, \Omega) \cong \mathrm{Sub}(A)$$

(natural in A), we obtain $\wedge_A$ that makes the following commute:

$$\begin{tikzcd}
\mathrm{Sub}(A)\times\mathrm{Sub}(A) \arrow[rr, "\cap"] \arrow[d, equal, "\wr"] & & \mathrm{Sub}(A) \arrow[dd, "\wr"] \\
\mathrm{Hom}(A,\Omega)\times\mathrm{Hom}(A,\Omega) \arrow[d, equal, "\wr"] & & \\
\mathrm{Hom}(A,\Omega\times\Omega) \arrow[rr, dashed, "\Lambda_A"'] & & \mathrm{Hom}(A,\Omega)
\end{tikzcd}$$

Then Λ_A is also natural in A so by Yoneda ($A := \Omega\times\Omega$, apply Λ_A to id), then Λ_A comes from a uniquely determined map

$$\Lambda : \Omega\times\Omega \to \Omega \qquad \text{(via composition)}$$

ie If subobjects S, T of A have char maps $s, t : A \to \Omega$, then $S \cap T$ has char map $A \xrightarrow{\langle s,t\rangle} \Omega\times\Omega \xrightarrow{\Lambda} \Omega$

And we will use the following for the next thm:

For a fixed ob. A, the op.

$$\cap : \mathrm{Sub}(A\times X)\times\mathrm{Sub}(A\times X) \to \mathrm{Sub}(A\times X) \quad \text{is } \underline{\text{natural}} \text{ in } X$$

By $\mathrm{Sub}(A\times X) \cong \mathrm{Hom}(X, PA)$, we get q:

$$\mathrm{Hom}(X, PA\times PA) \cong \mathrm{Hom}(X,PA)\times\mathrm{Hom}(X,PA) \xrightarrow{\Lambda_X} \mathrm{Hom}(X,PA)$$

(natural in X)

By Yoneda, we get a corr. map

$$\Lambda : PA\times PA \to PA$$

This is natural in A: $\forall\, k : B \to A$, we have

$$\begin{tikzcd}
PA\times PA \arrow[r, "\Lambda"] \arrow[d, "Pk\times Pk"'] & PA \arrow[d, "Pk"] \\
PB\times PB \arrow[r, "\Lambda"'] & PB
\end{tikzcd} \qquad \text{commutes}$$

CLAIM: $\mathrm{Sub}_{\mathcal{E}}(1)$ has exponentials

$U \in \mathrm{Ob}(\mathcal{E})$ s.t. $U \rightarrowtail 1$ is monic

- The exponential U^V of two $\boxed{\text{open}}$ objects $U \rightarrowtail 1$ and $V \rightarrowtail 1$ is again open.

  $\rightsquigarrow$ $U^V \hookrightarrow 1$ is monic by universal property of the exponential.

$\rightsquigarrow$ $\mathrm{Sub}_{\mathcal{E}}(1)$ has exponentials

SLICE: $\mathrm{Sub}_{\mathcal{E}}(A) \cong \mathrm{Sub}_{\mathcal{E}/A}(1)$

$\rightsquigarrow$ $\mathrm{Sub}_{\mathcal{E}}(A)$ is a Heyting algebra with the exponential in $\mathcal{E}/A$ as its implication operator.

The structure is natural in A:

- Let $k: A \to B$ be a morphism in $\mathcal{E}$. And consider the comm. square:

$$\begin{tikzcd}
\mathrm{Sub}_{\mathcal{E}}(B) \arrow[r, "k^{-1}"] \arrow[d, "i_B"'] & \mathrm{Sub}_{\mathcal{E}}(A) \arrow[d, "i_A"] \\
\mathcal{E}/B \arrow[r, "k^*"'] & \mathcal{E}/A
\end{tikzcd}$$

i_A: inclusion that identifies subobjects of A in $\mathcal{E}$ with subobjects of 1 in $\mathcal{E}/A$

Thm IV.7.2 (Thm 2 of previous section)

By thm in "slice of topos is a topos", then $\boxed{k^*}$ preserves exponentials, sums, and epimorphisms.

By lattice structure of $\mathrm{Sub}_{\mathcal{E}}(B)$ and $\mathrm{Sub}_{\mathcal{E}}(A)$, then k^{-1} is a Heyting alg. homomorphism. $\square$

Thm 2 (INTERNAL)

Let $A \in Ob(\mathcal{E})$.

The power object PA is an internal Heyting algebra.
(ie, the subobject classifier $\Omega := P1$ is an internal Heyting algebra).

The structure is natural in A:

ie Let $k : A \to B$ be a morphism in $\mathcal{E}$.
Then, the induced map $Pk : PB \to PA$ is a Heyting algebra homomorphism.

For each $X \in Ob(\mathcal{E})$, the internal structure on PA makes $Hom(X, PA)$ an external Heyting algebra so that the canonical isomorphism

$$Sub_{\mathcal{E}}(A \times X) \cong Hom_{\mathcal{E}}(X, PA)$$

is an isomorphism of external Heyting algebras.

What do we NTS?

- PA is a lattice object in $\mathcal{E}$: There are meet + join arrows s.t. equational identities hold.
- There is a binary op. $\Rightarrow : PA \times PA \to PA$

$\boxed{\text{proof}}$

The proof starts with some ideas stated in Section 6: "Factorizations and images":

And we will use the following for the next thm:

For a fixed ob. A, the op.

$$\cap : Sub(A \times X) \times Sub(A \times X) \longrightarrow Sub(A \times X)$$ is natural in X

By $Sub(A \times X) \cong Hom(X, PA)$, we get q:

$$Hom(X, PA \times PA) \cong Hom(X, PA) \times Hom(X, PA) \xrightarrow{\wedge_X} Hom(X, PA)$$

(natural in X)

By Yoneda, we get a corr. map

$$\wedge : PA \times PA \to PA$$

This is natural in A: $\forall\, k : B \to A$, we have

$$\begin{tikzcd} PA \times PA \arrow[r, "\wedge"] \arrow[d, "Pk \times Pk"'] & PA \arrow[d, "Pk"] \\ PB \times PB \arrow[r, "\wedge"'] & PB \end{tikzcd}$$

commutes

So we have

$$\wedge : PA \times PA \longrightarrow PA$$

where for any $X \in \mathcal{E}$, the meet operation on $\mathrm{Hom}_{\mathcal{E}}(X, PA)$ induced by composition corresponds to meet in the lattice

$$\mathrm{Sub}_{\mathcal{E}}(A \times X) \quad \text{via the iso:}$$

$$\mathrm{Hom}_{\mathcal{E}}(X, PA) \cong \mathrm{Sub}_{\mathcal{E}}(A \times X)$$

- Other ops are defined similarly.

eg $\boxed{\text{The } (\Rightarrow) \text{ operation}}$

For each $X \in \mathrm{Ob}(\mathcal{E})$, $\mathrm{Sub}_{\mathcal{E}}(A \times X)$ has an implication operator

$$\Rightarrow : \mathrm{Sub}_{\mathcal{E}}(A \times X) \times \mathrm{Sub}_{\mathcal{E}}(A \times X) \longrightarrow \mathrm{Sub}_{\mathcal{E}}(A \times X)$$

that is natural in X.

$\rightsquigarrow \exists! \Rightarrow_X$ (natural in X) such that the following commutes:

$$\begin{tikzcd}
\mathrm{Sub}_{\mathcal{E}}(A \times X) \times \mathrm{Sub}_{\mathcal{E}}(A \times X) \arrow[rr, "\Rightarrow"] \arrow[d, equal, "\wr"] & & \mathrm{Sub}_{\mathcal{E}}(A \times X) \arrow[dd, equal, "\wr"] \\
\mathrm{Hom}_{\mathcal{E}}(X, PA) \times \mathrm{Hom}_{\mathcal{E}}(X, PA) \arrow[d, equal, "\wr"] & & \\
\mathrm{Hom}_{\mathcal{E}}(X, PA \times PA) \arrow[rr, dashed, "\Rightarrow_X"'] & & \mathrm{Hom}_{\mathcal{E}}(X, PA)
\end{tikzcd}$$

By naturality of $\Rightarrow_X$ in X, then it must be induced (by Yoneda!) (via composition) by a uniquely determined map

$$\Rightarrow : PA \times PA \longrightarrow PA$$

- Top and bottom elements: similar

• $Pk: PB \to PA$ is a homomorphism of Heyting algebras:

By the naturality of the Heyting alg. structure on $\mathrm{Sub}_{\mathcal{E}}(A)$ and the comm. diagram:

$$\begin{tikzcd}
\mathrm{Sub}_{\mathcal{E}}(B \times X) \arrow[r, "\sim"] \arrow[d, "(k\times 1)^{-1}"'] & \mathrm{Hom}_{\mathcal{E}}(X, PB) \arrow[d, "\mathrm{Hom}_{\mathcal{E}}(X, Pk)"] \\
\mathrm{Sub}_{\mathcal{E}}(A \times X) \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathcal{E}}(X, PA)
\end{tikzcd}$$

Since $(k\times 1)^{-1}$ is a Heyting alg. hom, so is $\mathrm{Hom}_{\mathcal{E}}(X, Pk)$ for each X in $\mathcal{E}$.

$\Rightarrow$ Pk is a hom of internal Heyting algebras (by definitions) $\square$

Rem: Internal Heyting alg. structure on Ω is the unique one such that $\mathrm{Sub}_{\mathcal{E}}(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{E}}(X, \Omega)$ is a Heyting alg. isomorphism.