Cayley graphs and Fourier analysis on finite groups

Presentation notes

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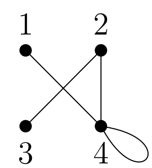
Presentation notes guideline:

- To be written on the board.
- To be said out loud.

Ex: A graph Γ is given by a vertex set V and an edge set E, e.g.,

$$V = \{1, 2, 3, 4\}$$
$$E = \{\{2, 3\}, \{1, 4\}, \{2, 4\}, \{4\}\}$$

Visually, we have:



A graph is finite if its vertex set V is finite. Running assumption in this section is that our graphs and groups are finite.

Def (5.4.1): Γ graph with $V = \{v_1, v_2, \ldots, v_n\}$ and E its edge set. The adjacency matrix of Γ (wrt V) is the $n \times n$-matrix $A = (a_{ij})_{i,j}$ where

$$a_{ij} = \begin{cases} 1, & \{v_i, v_j\} \in E \\ 0, & \text{else} \end{cases}$$

That is, there is a 1 in the (ij)-entry iff there exists an edge between vertex i and vertex j.

Ex: For the above graph,

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Note: Adjacency matrix is always symmetric. Since if $\{v_i, v_j\} \in E$, then $\{v_j, v_i\} \in E$ (undirected nature).

$\Rightarrow$ Always diagonalizable with real eigenvalues (spectral theorem).

Def: Set of eigenvalues of an adjacency matrix for a graph is called the spectrum of the graph.

Note: The spectrum does not depend on the ordering of the vertices.

For any relabelling σ of the vertices of a graph, the adjacency matrix is conjugate to a permutation matrix (corresponding to permuting the rows/columns according to σ).

Note: Spectrum of a graph reveals important information:

- Using the spectrum and the fact that A is always diagonalizable, we can easily compute the powers of A. $(A^n)_{ij}$ is the number of paths of length n from v_i to v_j.
- Spectral graph theory is an area of graph theory that studies graphs via their eigenvalues.
- Spectrum is used in random walks on graphs, Gerardo will be going into detail about this in the next presentation.

Given its significance, we can use representation theory to analyze the spectrum of a type of graph called a Cayley graph for abelian groups.

Goal: To describe the spectrum of a Cayley graph of an abelian group.

Def (5.4.3): A symmetric subset $S \subseteq G$ is a subset s.t.

- $e \notin S$; and
- $g \in S \Leftrightarrow g^{-1} \in S$

The Cayley graph of G wrt S, $\mathrm{Cay}(G, S)$, is the graph with:

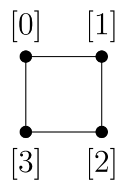
- Vertex set $V = G$;
- Edges: $\{g, h\} \in E \Leftrightarrow gh^{-1} \in S$.

Remark: S can be empty (i.e., no edges).

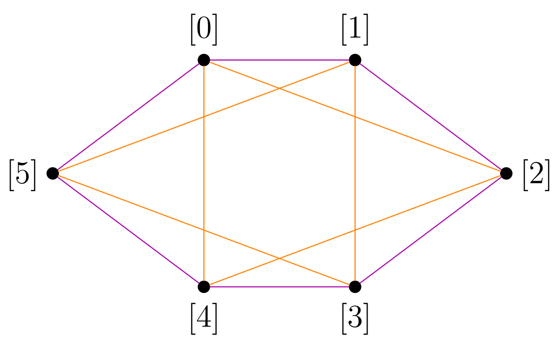
Fact: A Cayley graph is connected iff S generates G.

The graph is connected iff for every $x \in G$, x is connected to $e \in G$ by a path of edges,
iff every $x \in G$ can be written as $x = s_0 s_1 \cdots s_m$ for $s_i \in S$,
iff S generates G.

Ex (5.4.5):
$G = \mathbb{Z}/4\mathbb{Z}$, $S = \{[1],[3]\} = \{\pm[1]\}$.



Ex (5.4.6):
$G = \mathbb{Z}/6\mathbb{Z}$, $S = \{[1],[5],[2],[4]\} = \{\pm[1],\pm[2]\}$. (colour coding: $\pm[1]$, $\pm[2]$).



To describe the spectrum of a Cayley graph of an abelian group, we need a result about the group algebra $L(G)$.

Lemma (5.4.9): G abelian, $a \in L(G)$. Define $A : L(G) \to L(G)$ by

$$A(b) := a * b$$

Then, A is linear and for all $\chi \in \widehat{G}$, χ is an eigenvector of A with eigenvalue $\widehat{a}(\chi)$. i.e., A is a diagonalizable operator.

Pf:

As $(L(G), +, *)$ is a ring (Thm 5.2.3), A is linear. (by the distributivity of convolution over addition)

Let $n = |G|$, $\chi \in \widehat{G}$.

By Thm 5.3.8,

$$\begin{aligned} \widehat{a * \chi} &= \widehat{a} \cdot \widehat{\chi} \\ &= \widehat{a} \cdot n\delta_\chi \qquad [\widehat{\chi} = |G|\delta_\chi \text{ by ex. 5.3.5}] \end{aligned}$$

So $\forall\, \theta \in \widehat{G}$,

$$\begin{aligned} (\widehat{a} \cdot n\delta_\chi)(\theta) &= \begin{cases} \widehat{a}(\theta)n, & \chi = \theta \\ 0, & \text{else} \end{cases} \\ &= \widehat{a}(\chi)n\delta_\chi \end{aligned}$$

By inverse of Fourier transform, then

$$\begin{aligned} a * \chi &= \frac{1}{|G|} \sum_{\theta \in \widehat{G}} \widehat{a * \chi}(\theta)\theta \qquad \text{(Fourier inverse – Thm 5.3.6)} \\ &= \frac{1}{|G|} \sum_{\theta} \widehat{a}(\chi)n\delta_\chi(\theta)\theta \qquad \text{(by above computations)} \\ &= \frac{1}{n}\widehat{a}(\chi)n\chi \qquad (\text{by } \delta_\chi) \\ &= \widehat{a}(\chi)\chi \end{aligned}$$

That is,

$$A\chi = \widehat{a}(\chi)\chi$$

So χ is an eigenvector with eigenvalue $\widehat{a}(\chi)$.

As elements of $\widehat{G}$ form an orthonormal basis of eigenvectors for A, then A is diagonalizable.

- By 4.4.7, irreducible characters form orthonormal basis of $Z(L(G))$, so form a basis for space A is acting on (as G is abelian $Z(L(G)) = L(G)$.)

□

This lemma gives us a key step in computing the spectrum of a Cayley graph of an abelian group. It remains to express the adjacency matrix as the matrix of a convolution operator, so that we can apply this lemma directly.

Thm (5.4.10): $G = \{g_1, g_2, \ldots, g_n\}$ abelian group, $S \subseteq G$ symmetric.

$\chi_1, \chi_2, \ldots, \chi_n$ irred characters of G.

A the adjacency matrix of the Cayley graph of G wrt S (using above ordering for vertices).

Then:

(1) The eigenvalues of A are the numbers

$$\lambda_i = \sum_{s \in S} \chi_i(s) \quad \forall\, 1 \leq i \leq n$$

(2) The corresponding orthonormal basis of eigenvectors is given by v_i where

$$v_i = \frac{1}{\sqrt{|G|}} \begin{pmatrix} \chi_i(g_1) \\ \chi_i(g_2) \\ \vdots \\ \chi_i(g_n) \end{pmatrix}$$

Pf:

Define $\delta_S : G \to \mathbb{C}$ by

$$\delta_S := \sum_{s \in S} \delta_s \qquad \text{so that}$$

$$\delta_S(x) = \begin{cases} 1, & x \in S \\ 0, & \text{else} \end{cases}$$

Define $F : L(G) \to L(G)$ by

$$F(b) := \delta_S * b$$

That is, F is the convolution operator wrt δ_S.

By 5.4.9, irred characters χ_i are eigenvectors of F wrt eigenvalue $\widehat{\delta_S}(\chi_i)$, so

$$\begin{aligned} \lambda_i = \widehat{\delta_S}(\chi_i) &= |G| \langle \delta_S, \chi_i \rangle \qquad \text{by def of Fourier transform} \\ &= \frac{|G|}{|G|} \sum_{x \in S} \delta_S(x) \overline{\chi_i(x)} \\ &= \sum_{s \in S} \overline{\chi_i(s)} \qquad \text{by def of } \delta_S \\ &= \sum_{s \in S} \chi_i(s^{-1}) \qquad \text{since } \chi_i(g^{-1}) = \overline{\chi_i(g)} \text{ (prop 9.1.5)} \end{aligned}$$

$$= \sum_{s \in S} \chi_i(s) \qquad \text{as } S \text{ is symmetric}$$

Let B be the basis of $L(G)$ given by $B = \{\delta_{g_1}, \delta_{g_2}, \dots, \delta_{g_n}\}$.

For $i = 1, \dots, n$, $[F]_B$ has eigenvalues λ_i with eigenvectors v_i.

- As λ_i are orthonormal, then v_i are orthonormal.
- The scaling by $\frac{1}{\sqrt{|G|}}$ comes from the fact that the δ_{g_i} are othonormal wrt the inner product $(f_1, f_2) = |G|\langle f_1, f_2 \rangle$.

Remains to show that the adjacency matrix $A = [F]_B$.

Let $\delta_{g_j} \in B$.

$$F(\delta_{g_j}) = \sum_{s \in S} \delta_s * \delta_{g_j} \qquad \text{by def and } \delta_S$$
$$= \sum_{s \in S} \delta_{sg_j} \qquad \text{by prop 5.2.2, } \delta_s * \delta_{g_j} = \delta_{sg_j}$$

The ij-entry of $[F]_B$ is the coefficient of δ_{g_i} in $F(\delta_{g_j}) = \sum_{s \in S} \delta_{sg_j}$. Thus,

$$([F]_B)_{ij} = \begin{cases} 1, & g_i = sg_j \text{ for some } s \in S \\ 0, & \text{else} \end{cases}$$
$$= \begin{cases} 1, & g_i g_j^{-1} \in S \\ 0, & \text{else} \end{cases}$$
$$= A_{ij} \qquad \text{for all } i, j.$$

Note that by the spectral theorem, $\lambda_i \in \mathbb{R}$ for all i. □

A direct result of this theorem is applying it to a Cayley graph of a cyclic group.

Cor (5.4.11): Let A be the adjacency matrix of the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ wrt the symmetric set S. The eigenvalues of A are

$$\lambda_k = \sum_{[m] \in S} e^{\frac{2\pi i k m}{n}} \qquad k = 0, 1, \dots, n-1$$

The corresponding eigenvectors are given by Lemma 5.4.9.

Pf: A straight application of 5.4.10. □

Ex (5.4.12):

Let A be the adjacency matrix of the graph in Ex 5.4.6:

$G = \mathbb{Z}/6\mathbb{Z}$, $S = \{\pm[1], \pm[2]\}$.

For $k = 0, \ldots, 5$, the eigenvalues of A are

$$\begin{aligned}\lambda_k &= \sum_{[m]\in S} e^{\frac{2\pi i k m}{6}} \\ &= e^{\frac{\pi i k}{3}} + e^{\frac{-\pi i k}{3}} + e^{\frac{2\pi i k}{3}} + e^{\frac{-2\pi i k}{3}} \\ &= 2\cos\left(\frac{\pi k}{3}\right) + 2\cos\left(\frac{2\pi k}{3}\right)\end{aligned}$$

Remark: This approach can be generalized to non-abelian groups, with the assumption that S is closed under conjugation.

In particular, each character χ induces eigenvalue(s) for A as follows.

Prop: G finite, $S \subseteq G$ symmetric and invariant under conjugation. Then, for each $\chi_i \in \widehat{G}$, the adjacency matrix of $\mathrm{Cay}(G, S)$ has eigenvalue

$$\lambda_i = \frac{1}{\chi_i(e)} \sum_{s\in S} \chi_i(s)$$

with multiplicity $\chi_i(e)^2 = d_i^2$.