01.04.2021

Reedy categories and diagrams

Motivation:

Goal: To introduce the relevant terminology and ideas to understand the proof of the model category structure on Reedy diagrams.

Sources:
- "Model categories and their localizations" [Hirschhorn]
- "A primer on homotopy colimits" [Dugger]

§ 1. Preliminaries

$\boxed{\text{Def}}$ $\mathcal{C}$ a small category.

$\mathcal{C}$ is a **REEDY CATEGORY** if it has two wide subcategories $\vec{\mathcal{C}}$ and $\overleftarrow{\mathcal{C}}$ where each object is assigned a "**DEGREE**" $n \geq 0$ such that the following is satisfied:

(1) Every non-identity morphism of $\vec{\mathcal{C}}$ raises degree.
(2) Every non-identity morphism of $\overleftarrow{\mathcal{C}}$ lowers degree.
(3) Every morphism g of $\mathcal{C}$ has a unique factorization

$$g = \vec{g}\,\overleftarrow{g} \qquad \vec{g} \in \vec{\mathcal{C}},\ \overleftarrow{g} \in \overleftarrow{\mathcal{C}}$$

Remark:

- More general definition: degree function takes ORDINAL values

Some basic properties:

- $\mathcal{C}$ Reedy $\Rightarrow$ $\mathcal{C}^{op}$ is Reedy with $\overrightarrow{\mathcal{C}^{op}} := (\overleftarrow{\mathcal{C}})^{op}$ and $\overleftarrow{\mathcal{C}^{op}} := (\vec{\mathcal{C}})^{op}$
- $\mathcal{C}, \mathcal{D}$ Reedy $\Rightarrow$ $\mathcal{C} \times \mathcal{D}$ is Reedy with $\overrightarrow{\mathcal{C} \times \mathcal{D}} := \vec{\mathcal{C}} \times \vec{\mathcal{D}}$, $\overleftarrow{\mathcal{C} \times \mathcal{D}} := \overleftarrow{\mathcal{C}} \times \overleftarrow{\mathcal{D}}$

Example: The cosimplicial and simplicial indexing categories.

$\boxed{\text{Def}}$ $n \in \mathbb{Z}_{\geq 0}$

$[n] := \{0, 1, \ldots, n\}$

cat $\Delta := \begin{cases} \text{objects: } \{[n] \mid n \geq 0\} \\ \text{morphisms: } \Delta([n],[k]) := \{ \delta : [n] \to [k] \mid \delta(i) \leq \delta(j) \ \forall\, 0 \leq i \leq j \leq n \} \end{cases}$

- Δ is called the COSIMPLICIAL INDEXING CATEGORY
- Δ^{op} is called the SIMPLICIAL INDEXING CATEGORY

- $\mathcal{M}$ a category
  - $\rightsquigarrow$ Functor $\Delta^{op} \to \mathcal{M}$: A SIMPLICAL OBJECT IN M
  - $\rightsquigarrow$ Functor $\Delta \to M$: A COSIMPLICIAL OBJECT in M

Notation:

- X a simplicial object in M $\rightsquigarrow$ denote $X_{[n]} =: X_n$
- X a cosimplicial object in M $\rightsquigarrow$ denote $X_{[n]} =: X^n$

These are Reedy categories!

- Cosimplicial indexing category Δ

with $\deg([n]) := n$

$\vec{\Delta} := \{\text{injective maps}\}$

$\overleftarrow{\Delta} := \{\text{surjective maps}\}$

- Simplicial indexing category Δ^{op}

with $\deg([n]) := n$

$\overrightarrow{\Delta^{op}} := \{\text{op}^s \text{ of surjective maps}\}$

$\overleftarrow{\Delta^{op}} := \{\text{op}^s \text{ of injective maps}\}$

§ 2. Filtrations

$\boxed{\text{Def}}$ • $\mathcal{C}$ is Reedy

• $n \in \mathbb{Z}_{\geq 0}$

• The **n-FILTRATION** $F^n\mathcal{C}$ is the full subcategory of $\mathcal{C}$ with all objects $\alpha \in \mathcal{C}$ s.t. $\deg(\alpha) \leq n$

eg 0-filtration of $\mathcal{C}$ = $\begin{cases} \text{obj} : \alpha \ \text{ s.t. } \deg(\alpha) = 0 \\ \text{mor} : \{1_\alpha \mid \deg(\alpha) = 0\} \end{cases}$ ← Reedy factorization

$\boxed{\text{Proposition}}$ $\mathcal{C}$ Reedy. $n \in \mathbb{Z}_{\geq 0}$.

$\Rightarrow$ $F^n\mathcal{C}$ is Reedy with:

• $\overrightarrow{F^n\mathcal{C}} := \overrightarrow{\mathcal{C}} \cap (F^n\mathcal{C})$

• $\overleftarrow{F^n\mathcal{C}} := \overleftarrow{\mathcal{C}} \cap (F^n\mathcal{C})$

and

$\mathcal{C} = \bigcup_{n \geq 0} F^n\mathcal{C}$ where $F^0\mathcal{C} \subset F^1\mathcal{C} \subset F^2\mathcal{C} \subset \cdots$

§ 3. Diagrams

In this section: $\mathcal{C}$ is a Reedy category, $\mathcal{M}$ is a model category

<u>Recall</u>: The 0-filtration of a Reedy category has no non-identity maps.

$\rightsquigarrow$ define a diagram $X : F^0\mathcal{C} \to \mathcal{M}$ by choosing an object $X_\alpha \in \mathcal{M}$ for each object $\alpha \in \mathcal{C}$ with $\deg \alpha = 0$.

Suppose we have $X : F^{n-1}\mathcal{C} \longrightarrow \mathcal{M}$

want to extend it to a diagram $X : F^n\mathcal{C} \to \mathcal{M}$

Define $X : F^n\mathcal{C} \to M$ by:

① Objects

For each $\alpha \in \mathcal{C}$ with $\deg \alpha = n$, choose $X_\alpha \in M$.

② Morphisms

For each $\beta \in F^{n-1}\mathcal{C}$, and $\boxed{\beta \to \alpha}$ in $F^n\mathcal{C}$,

(*) Want a map $X_\beta \longrightarrow X_\alpha$ s.t. if $\beta \to \beta' \in F^{n-1}\mathcal{C}$ and

$$\begin{tikzcd} \beta \arrow[rr] \arrow[dr] & & \beta' \arrow[dl] \\ & \alpha & \end{tikzcd}$$ commutes in $F^n\mathcal{C}$

$$\Downarrow$$

$$\Rightarrow \begin{tikzcd} X_\beta \arrow[rr] \arrow[dr] & & X_{\beta'} \arrow[dl] \\ & X_\alpha & \end{tikzcd}$$ commutes in M.

Let $i^n : F^{n-1}\mathcal{C} \to F^n\mathcal{C}$ be the inclusion functor.

Then, (*) is equivalent to choosing a map

$$\mathrm{Lan}_{i^n} X(\alpha) = \operatorname{colim}_{(i^n \downarrow \alpha)} X \longrightarrow X$$

Dually, for each $\gamma \in F^{n-1}\mathcal{C}$ and $\boxed{\alpha} \to \gamma$ in $F^n\mathcal{C}$,

we need a map $X_\alpha \to X_\gamma$ such that

If $$\boxed{\begin{tikzcd} & \alpha \arrow[dl] \arrow[dr] & \\ \gamma \arrow[rr] & & \gamma' \end{tikzcd}}$$ commutes in $F^n\mathcal{C}$, then

$$\Downarrow$$

$$\begin{tikzcd} & X_\alpha \arrow[dl] \arrow[dr] & \\ X_\gamma \arrow[rr] & & X_{\gamma'} \end{tikzcd}$$ commutes in M.

This is equivalent to choosing a map

$$X_\alpha \longrightarrow \lim_{(\alpha \downarrow i^n)} X = \mathrm{Ran}_{i^n} X(\alpha)$$

Key: The colimit and limit chosen above are independent of our choice of degree function on the Reedy category $\mathcal{C}$.

BUT, we need some conditions on the choices for these maps.

If $(\beta \to \gamma) \in F^{n-1}\mathcal{C}$ and

$$\begin{tikzcd} & \alpha \arrow[dr] & \\ \beta \arrow[ur] \arrow[rr] & & \gamma \end{tikzcd}$$

commutes in $F^n\mathcal{C}$

$$\Rightarrow \quad \begin{tikzcd} & X_\alpha \arrow[dr] & \\ X_\beta \arrow[ur] \arrow[rr] & & X_\gamma \end{tikzcd}$$

must commute in M.

Equivalent to:

$$\begin{tikzcd} & X_\alpha \arrow[dr] & \\ \operatorname{colim}_{(i^n \downarrow \alpha)} X \arrow[ur] \arrow[rr] & & \lim_{(\alpha \downarrow i^n)} X \end{tikzcd} \qquad (*)$$

• **Theorem:** $(*)$ is enough to construct an extension of X from $F^{n-1}\mathcal{C}$ to $F^n\mathcal{C}$.

(Exercise)

§ 4. Latching and matching objects

$\alpha = [n]$

$\mathcal{C}$ a Reedy category, $\alpha \in \mathcal{C}$ an object.

M a model category, $X : \mathcal{C} \to M$

Def • LATCHING CATEGORY OF $\mathcal{C}$ at α, denoted $\partial(\vec{\mathcal{C}}\downarrow\alpha)$, is the full subcategory of $(\vec{\mathcal{C}}\downarrow\alpha)$ containing all the objects except 1_α.

• MATCHING CATEGORY OF $\mathcal{C}$ at α, denoted $\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})$ is the full subcategory of $(\alpha\downarrow\overleftarrow{\mathcal{C}})$ containing all the objects except 1_α.

Facts: ① $\left(\partial(\vec{\mathcal{C}}\downarrow\alpha)\right)^{op} \cong \partial(\alpha\downarrow\overleftarrow{\mathcal{C}^{op}})$

② $\left(\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})\right)^{op} \cong \partial(\overrightarrow{\mathcal{C}^{op}}\downarrow\alpha)$

$$\partial(\vec{\mathcal{C}}\downarrow\alpha) \overset{i}{\hookrightarrow} (\vec{\mathcal{C}}\downarrow\alpha)$$

Def • LATCHING OBJECT OF X at α: $L_\alpha X := \underset{\partial(\vec{\mathcal{C}}\downarrow\alpha)}{\operatorname{colim}} X = \operatorname{Lan}_i X(\alpha)$

- LATCHING MAP OF X at α: $L_\alpha X \to X$

• MATCHING OBJECT OF X at α: $M_\alpha X := \underset{\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})}{\lim} X = \operatorname{Ran}_i X(\alpha)$

- MATCHING MAP OF X at α: $X_\alpha \to M_\alpha X$

Def Suppose $\alpha \in \mathcal{C}$ has degree n.

• $\partial(\alpha\downarrow F^n\mathcal{C}) := \left\{ \begin{array}{l} \text{full subcategory of } (\alpha\downarrow F^n\mathcal{C}) \text{ with objects the maps} \\ \alpha \xrightarrow{g} \beta \text{ s.t. } \exists \text{ a factorization } \alpha \xrightarrow{\overleftarrow{g}} \gamma \xrightarrow{\vec{g}} \beta \text{ with } \overleftarrow{g} \in \overleftarrow{\mathcal{C}},\ \vec{g} \in \vec{\mathcal{C}} \\ \text{and } \overleftarrow{g} \neq 1_\alpha \end{array} \right\}$

• $\partial(F^n\mathcal{C}\downarrow\alpha) := \left\{ \begin{array}{l} \text{full subcategory of } (F^n\mathcal{C}\downarrow\alpha) \text{ with objects the maps } \beta \xrightarrow{g} \alpha \\ \text{s.t. } \exists \text{ factorization } \beta \xrightarrow{\overleftarrow{g}} \gamma \xrightarrow{\vec{g}} \alpha \text{ with } \overleftarrow{g} \in \overleftarrow{\mathcal{C}},\ \vec{g} \in \vec{\mathcal{C}}, \text{ and } \vec{g} \neq 1_\alpha \end{array} \right\}$

objects used to analyze maps between Reedy diagrams: $\operatorname{colim}_{\partial(F^n\mathcal{C}\downarrow\alpha)} X$ and $\lim_{\partial(\alpha\downarrow F^n\mathcal{C})} X$

Key fact: All colim are latching objects of X

All lim are matching objects of X.

$X : F^1\mathcal{C} \to M$

$[1]\to[2]$,

$[0]\to[2]$

$M_2 X =$

- If $\alpha\in\mathcal{C}$ has degree n, $i^n : F^{n-1}\mathcal{C}\to F^n\mathcal{C}$ the inclusion functor.

(1) The LATCHING CATEGORY $\partial(\vec{\mathcal{C}}\downarrow\alpha)$ is a right cofinal subcat. of both $(i^n\downarrow\alpha)$ and $\partial(F^n\mathcal{C}\downarrow\alpha)$

(2) The MATCHING CATEGORY $\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})$ is a left cofinal subcat. of both $(\alpha\downarrow i^n)$ and $\partial(\alpha\downarrow F^n\mathcal{C})$

$\mathcal{C}, M, \alpha\in\mathcal{C}$ as above.

Proposition (*) Let $X : \mathcal{C}\to M$ (a $\mathcal{C}$-diagram in M), $i^n : F^{n-1}\mathcal{C}\to F^n\mathcal{C}$ the inclusion functor. There are natural isomorphisms:

$$L_\alpha X \cong \operatorname*{colim}_{(i^n\downarrow\alpha)} X \cong \operatorname*{colim}_{\partial(F^n\mathcal{C}\downarrow\alpha)} X$$

And

$$M_\alpha X \cong \lim_{(\alpha\downarrow i^n)} X \cong \lim_{\partial(\alpha\downarrow F^n\mathcal{C})} X$$

Summary: What do we have?

- $\mathcal{C}$ Reedy, M a model category
- $X : F^{n-1}\mathcal{C} \to M$ diagram indexed by $(n-1)$-filtration of $\mathcal{C}$.
- $\alpha \in \mathcal{C}$ with deg n.

$$\begin{tikzcd} & X_\alpha \arrow[dr] & \\ \operatorname{colim} X \, (i^n \downarrow \alpha) \arrow[ur] \arrow[rr] & & \lim X \, (\alpha \downarrow i^n) \end{tikzcd}$$

$\rightsquigarrow$ There is a natural map $L_\alpha X \longrightarrow M_\alpha X$

$\rightsquigarrow$ Extending X to a diagram $F^n\mathcal{C} \to M$ is equivalent to the following:

For each $\alpha \in \mathcal{C}$ of deg n, choose an object X_α and a factorization

$$L_\alpha X \to X_\alpha \to M_\alpha X$$

of the natural map.

$$\begin{tikzcd} & X_\alpha & \\ L_\alpha X \arrow[ur] \arrow[rr] & & M_\alpha X \arrow[ul] \end{tikzcd}$$

- This can be done independently for each object of deg n

§ 5. Maps between Reedy diagrams

Same situation as above: $\mathcal{C}$, M, $\alpha \in \mathcal{C}$, $\deg \alpha = n$

- $X, Y : \mathcal{C} \to M$ diagrams

Goal: Define $X \Rightarrow Y$ inductively on filtrations.

Recall: $F^0\mathcal{C}$ contains no non-identity maps.

$\rightsquigarrow$ $f : X|_{F^0\mathcal{C}} \Rightarrow Y|_{F^0\mathcal{C}}$ is completely determined by choosing a map $X_\alpha \longrightarrow Y_\alpha$ for every object $\alpha \in \mathcal{C}$ with degree 0.

Suppose $f : X|_{F^{n-1}\mathcal{C}} \Rightarrow Y|_{F^{n-1}\mathcal{C}}$

For every $\alpha \in \mathcal{C}$ with $\deg \alpha = n$, we have:

$$
\begin{tikzcd}
\operatorname{colim}\limits_{(i^n \downarrow \alpha)} X \arrow[r] \arrow[d] & X_\alpha \arrow[r] & \lim\limits_{(\alpha \downarrow i^n)} X \arrow[d] \\
\operatorname{colim}\limits_{(i^n \downarrow \alpha)} Y \arrow[r] & Y_\alpha \arrow[r] & \lim\limits_{(\alpha \downarrow i^n)} Y
\end{tikzcd}
$$

Proposition (*) $\rightsquigarrow$

$$
\begin{tikzcd}
L_\alpha X \arrow[r] \arrow[d] & X_\alpha \arrow[r] \arrow[d, dashed] & M_\alpha X \arrow[d] \\
L_\alpha Y \arrow[r] & Y_\alpha \arrow[r] & M_\alpha Y
\end{tikzcd}
$$

$$\left\{ \text{Extensions of } f \text{ to } F^n C \right\} \xleftrightarrow{\text{correspondence}} \left\{ \begin{array}{l} \text{Choice, for each } \alpha \in \mathcal{C} \text{ with deg } n, \text{ of a} \\ \text{dashed arrow that makes squares commute} \end{array} \right\}$$

Suppose $A, B, X, Y : \mathcal{C} \to M$ are Reedy diagrams and the following commutes:

(1)

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d] \\
B \arrow[r] \arrow[ur, dotted, "h"] & Y
\end{tikzcd}
$$

where h is only defined on $B|_{F^{n-1}\mathcal{C}}$.

Then, for every $\alpha \in C$ with deg n, we have an induced diagram:

$$(2) \qquad \begin{tikzcd} A_\alpha \coprod_{L_\alpha A} L_\alpha B \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ B_\alpha \arrow[r] \arrow[ur, dashed, "h_\alpha"] & Y_\alpha \times_{M_\alpha Y} M_\alpha X \end{tikzcd}$$

$\{$ h can be extended to $F^n C$ so that (1) commutes $\}$

$\updownarrow$ correspondence

$\{$ for every $\alpha \in C$ of deg n, there is a map $B_\alpha \xrightarrow{h_\alpha} X_\alpha$ so that (2) commutes $\}$

- Have the ability to define the Reedy model structure.

$\Rightarrow$ <u>statement and proof</u>: next week featuring Sayantan!