

Reedy categories and diagrams

Motivation:

Goal: To introduce the relevant terminology and ideas to understand the proof of the model category structure on Reedy diagrams.

Sources:

- "Model categories and their localizations" [Hirschhorn]
- "A primer on homotopy colimits" [Dugger]

§ 1. Preliminaries

Def \mathcal{C} a small category.

\mathcal{C} is a REEDY CATEGORY if it has two wide subcategories $\vec{\mathcal{C}}$ and $\overleftarrow{\mathcal{C}}$ where each object is assigned a "DEGREE" $n \geq 0$ such that the following is satisfied:

- $$\left\{ \begin{array}{l} (1) \text{ Every non-identity morphism of } \vec{\mathcal{C}} \text{ raises degree.} \\ (2) \text{ Every non-identity morphism of } \overleftarrow{\mathcal{C}} \text{ lowers degree.} \end{array} \right.$$

(3) Every morphism g of \mathcal{C} has a unique factorization

$$g = \vec{g} \overleftarrow{g} \quad \vec{g} \in \vec{\mathcal{C}}, \overleftarrow{g} \in \overleftarrow{\mathcal{C}}$$

Remark:

- More general definition: degree function takes ORDINAL values

Some basic properties:

- \mathcal{C} Reedy $\Rightarrow \mathcal{C}^{\mathcal{P}}$ is Reedy with $\vec{\mathcal{C}^{\mathcal{P}}} := (\vec{\mathcal{C}})^{\mathcal{P}}$ and $\overleftarrow{\mathcal{C}^{\mathcal{P}}} := (\overleftarrow{\mathcal{C}})^{\mathcal{P}}$
- \mathcal{C}, \mathcal{D} Reedy $\Rightarrow \mathcal{C} \times \mathcal{D}$ is Reedy with $\vec{\mathcal{C} \times \mathcal{D}} := \vec{\mathcal{C}} \times \vec{\mathcal{D}}$ and $\overleftarrow{\mathcal{C} \times \mathcal{D}} := \overleftarrow{\mathcal{C}} \times \overleftarrow{\mathcal{D}}$

Example: The cosimplicial and simplicial indexing categories.

Def $n \in \mathbb{Z}_{\geq 0}$

$$[n] := \{0, 1, \dots, n\}$$

cat $\Delta := \begin{cases} \text{objects: } \{[n] \mid n \geq 0\} \\ \text{morphisms: } \Delta([n], [k]) := \{ \sigma: [n] \rightarrow [k] \mid \sigma(i) \leq \sigma(j) \ \forall 0 \leq i \leq j \leq n \} \end{cases}$

• Δ is called the COSIMPLICIAL INDEXING CATEGORY

• Δ^{op} is called the SIMPLICIAL INDEXING CATEGORY

• M a category

\rightsquigarrow Functor $\Delta^{\text{op}} \rightarrow M$: A SIMPLICIAL OBJECT IN M

\rightsquigarrow Functor $\Delta \rightarrow M$: A COSIMPLICIAL OBJECT in M

Notation :

• X a simplicial object in M \rightsquigarrow denote $X_{[n]} =: X_n$

• X a cosimplicial object in M \rightsquigarrow denote $X_{[n]} =: X^n$

These are Reedy categories!

• Cosimplicial indexing category Δ

with $\deg([n]) := n$

$$\vec{\Delta} := \{ \text{injective maps} \}$$

$$\overleftarrow{\Delta} := \{ \text{surjective maps} \}$$

• Simplicial indexing category Δ^{op}

with $\deg([n]) := n$

$$\overrightarrow{\Delta^{\text{op}}} := \{ \text{opposite of surjective maps} \}$$

$$\overleftarrow{\Delta^{\text{op}}} := \{ \text{opposite of injective maps} \}$$

§ 2. Filtrations

Def

• \mathcal{C} is Reedy

• $n \in \mathbb{Z}_{\geq 0}$

• The n -FILTRATION $(F^n \mathcal{C})$ is the full subcategory of \mathcal{C} with all objects $\alpha \in \mathcal{C}$ s.t. $\deg(\alpha) \leq n$

eg 0-filtration of $\mathcal{C} = \left\{ \begin{array}{l} \underline{\text{obj}} : \alpha \text{ s.t. } \deg(\alpha) = 0 \\ \underline{\text{mor}} : \{ \alpha \mid \deg(\alpha) = 0 \} \end{array} \right\} \leftarrow \text{Reedy Factorization}$

Proposition

\mathcal{C} Reedy. $n \in \mathbb{Z}_{\geq 0}$.

$\Rightarrow F^n \mathcal{C}$ is Reedy with:

$$\cdot \overrightarrow{F^n \mathcal{C}} := \overrightarrow{\mathcal{C}} \cap (F^n \mathcal{C})$$

$$\cdot \overleftarrow{F^n \mathcal{C}} := \overleftarrow{\mathcal{C}} \cap (F^n \mathcal{C})$$

and

$$\mathcal{C} = \bigcup_{n \geq 0} F^n \mathcal{C} \quad \text{where} \quad F^0 \mathcal{C} \subset F^1 \mathcal{C} \subset F^2 \mathcal{C} \subset \dots$$

§ 3. Diagrams

In this section: \mathcal{C} is a Reedy category, M is a model category

Recall: The 0-filtration of a Reedy category has no non-identity maps.

\leadsto define a diagram $X: F^0 \mathcal{C} \rightarrow M$ by choosing an object $X_\alpha \in M$ for each object $\alpha \in \mathcal{C}$ with $\deg \alpha = 0$.

Suppose we have $X: (F^{n-1} \mathcal{C}) \rightarrow M$

want to extend it to a diagram $X: F^n \mathcal{C} \rightarrow M$

Define $X: F^n \mathcal{C} \rightarrow M$ by:

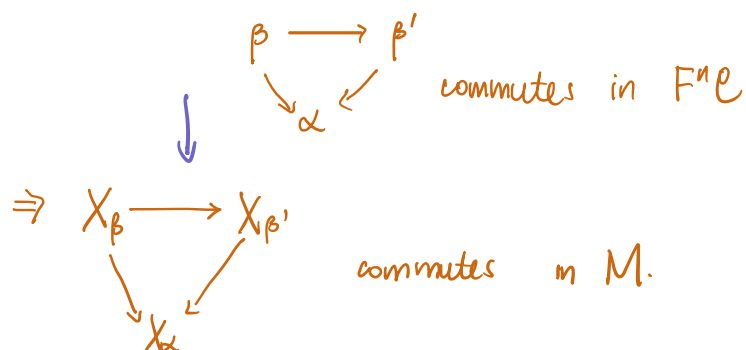
I. Objects

For each $\alpha \in \mathcal{C}$ with $\deg \alpha = n$, choose $X_\alpha \in M$.

II. Morphisms

For each $\beta \in F^{n-1} \mathcal{C}$, and $\boxed{\beta \rightarrow \alpha}$ in $F^n \mathcal{C}$,

(*) want a map $X_\beta \longrightarrow X_\alpha$ s.t. if $\beta \rightarrow \beta' \in F^{n-1} \mathcal{C}$ and



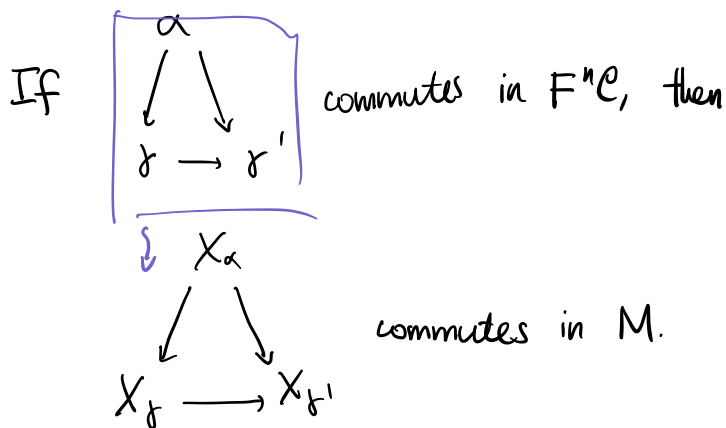
Let $i^n: F^{n-1} \mathcal{C} \rightarrow F^n \mathcal{C}$ be the inclusion functor.

Then, (*) is equivalent to choosing a map

$$\text{Lan}_{i^n} X(\alpha) = \text{colim}_{(i^n \downarrow \alpha)} X \longrightarrow X$$

Dually, for each $\gamma \in F^{n-1} \mathcal{C}$ and $\boxed{\alpha \rightarrow \gamma}$ in $F^n \mathcal{C}$,

we need a map $X_\alpha \rightarrow X_\gamma$ such that



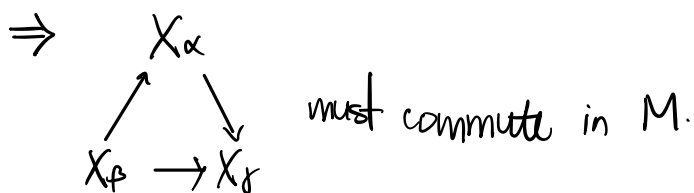
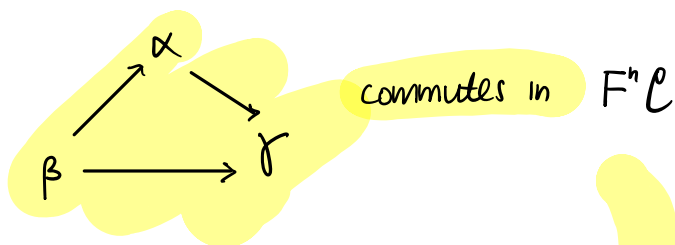
This is equivalent to choosing a map

$$X_\alpha \longrightarrow \lim_{(\alpha \downarrow i^n)} X = \text{Ran}_{i^n} X(\alpha)$$

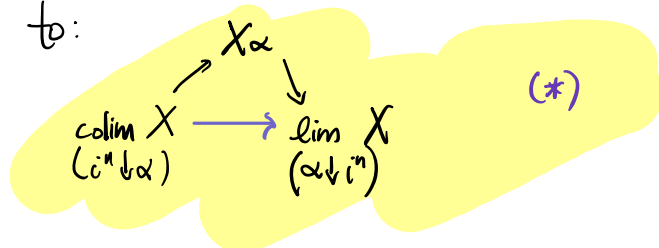
key: The colimit and limit chosen above are independent of our choice of degree function on the Reedy category \mathcal{C} .

BUT, we need some conditions on the choices for these maps.

If $(\beta \rightarrow \gamma) \in F^{n-1}\mathcal{C}$ and



Equivalent to:



• Theorem: $(*)$ is enough to construct an extension of X from $F^{n-1}\mathcal{C}$ to $F^n \mathcal{C}$.

(Exercise)

§ 4. Latching and matching objects

$$\alpha = [\cdot]$$

\mathcal{C} a Reedy category, $\alpha \in \mathcal{C}$ an object.

M a model category, $X: \mathcal{C} \rightarrow M$

- Def**
- LATCHING CATEGORY OF \mathcal{C} at α , denoted $\partial(\vec{\mathcal{C}} \downarrow \alpha)$, is the full subcategory of $(\vec{\mathcal{C}} \downarrow \alpha)$ containing all the objects except 1_α .
 - MATCHING CATEGORY OF \mathcal{C} at α , denoted $\partial(\alpha \downarrow \vec{\mathcal{C}})$ is the full subcategory of $(\alpha \downarrow \vec{\mathcal{C}})$ containing all the objects except 1_α .

- Facts:**
- $(\partial(\vec{\mathcal{C}} \downarrow \alpha))^{\text{op}} \cong \partial(\alpha \downarrow \vec{\mathcal{C}}^{\text{op}})$
 - $(\partial(\alpha \downarrow \vec{\mathcal{C}}))^{\text{op}} \cong \partial(\vec{\mathcal{C}}^{\text{op}} \downarrow \alpha)$

$$\partial(\vec{\mathcal{C}} \downarrow \alpha) \xrightarrow{i} (\vec{\mathcal{C}} \downarrow \alpha)$$

- Def**
- LATCHING OBJECT OF X at α : $L_\alpha X := \text{colim}_{\partial(\vec{\mathcal{C}} \downarrow \alpha)} X = \text{Lan}_i X(\alpha)$
 - LATCHING MAP OF X at α : $L_\alpha X \rightarrow X$

- MATCHING OBJECT OF X at α : $M_\alpha X := \lim_{\partial(\alpha \downarrow \vec{\mathcal{C}})} X = \text{Ran}_i X(\alpha)$
- MATCHING MAP OF X at α : $X_\alpha \rightarrow M_\alpha X$

Def Suppose $\alpha \in \mathcal{C}$ has degree n .

- $\partial(\alpha \downarrow F^n \mathcal{C}) := \left\{ \begin{array}{l} \text{full subcategory of } (\alpha \downarrow F^n \mathcal{C}) \text{ with objects the maps} \\ \alpha \xrightarrow{q} \beta \text{ s.t. } \exists \text{ a factorization } \alpha \xrightarrow{\tilde{q}} r \xrightarrow{\bar{q}} \beta \text{ with } \tilde{q} \in \vec{\mathcal{C}}, \bar{q} \in \vec{\mathcal{C}} \\ \text{and } \bar{q} \neq 1_\alpha \end{array} \right\}$
- $\partial(F^n \mathcal{C} \downarrow \alpha) := \left\{ \begin{array}{l} \text{full subcategory of } (F^n \mathcal{C} \downarrow \alpha) \text{ with objects the maps } \beta \xrightarrow{q} \alpha \\ \text{s.t. } \exists \text{ factorization } \beta \xrightarrow{\tilde{q}} r \xrightarrow{\bar{q}} \alpha \text{ with } \tilde{q} \in \vec{\mathcal{C}}, \bar{q} \in \vec{\mathcal{C}}, \text{ and } \bar{q} \neq 1_\alpha \end{array} \right\}$

objects used to analyze maps between Reedy diagrams: $\operatorname{colim}_{\partial(F^n \mathcal{C} \downarrow \alpha)} X$ and $\lim_{\partial(\alpha \downarrow F^n \mathcal{C})} X$

Key fact: All colim are latching objects of X

All \lim are matching objects of X .

$$X : F\mathcal{C} \rightarrow M$$

$$[1] \rightarrow [2], \\ [0] \rightarrow [2]$$

• If $\alpha \in \mathcal{C}$ has degree n , $i^n : F^{n-1}\mathcal{C} \rightarrow F^n\mathcal{C}$ the inclusion functor.

$$M_2 X =$$

(1) The LATCHING CATEGORY $\partial(\tilde{\mathcal{C}} \downarrow \alpha)$ is a right cofinal subcat. of both $(i^n \downarrow \alpha)$ and $\partial(F^n \mathcal{C} \downarrow \alpha)$

(2) The MATCHING CATEGORY $\partial(\alpha \downarrow \tilde{\mathcal{C}})$ is a left cofinal subcat. of both $(\alpha \downarrow i^n)$ and $\partial(\alpha \downarrow F^n \mathcal{C})$

$\mathcal{C}, M, \alpha \in \mathcal{C}$ as above.

Proposition Let $X : \mathcal{C} \rightarrow M$ (a \mathcal{C} -diagram in M), $i^n : F^{n-1}\mathcal{C} \rightarrow F^n\mathcal{C}$ the inclusion functor. There are natural isomorphisms:

(*)

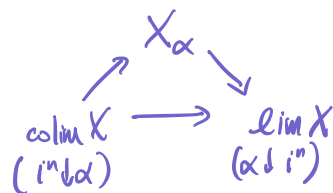
$$\begin{aligned} L_\alpha X &\cong \operatorname{colim}_{(i^n \downarrow \alpha)} X \\ &\cong \operatorname{colim}_{\partial(F^n \mathcal{C} \downarrow \alpha)} X \end{aligned}$$

And

$$\begin{aligned} M_\alpha X &\cong \lim_{(\alpha \downarrow i^n)} X \\ &\cong \lim_{\partial(\alpha \downarrow F^n \mathcal{C})} X \end{aligned}$$

Summary: What do we have?

- \mathcal{C} Reedy, M a model category
- $X: F^{\bullet}\mathcal{C} \rightarrow M$ diagram indexed by $(n-1)$ -filtration of \mathcal{C} .
- $\alpha \in \mathcal{C}$ with $\deg \alpha = n$.



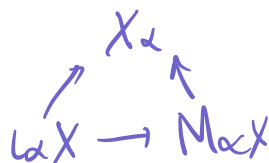
\leadsto There is a natural map $L_{\alpha}X \rightarrow M_{\alpha}X$

\leadsto Extending X to a diagram $F^{\bullet}\mathcal{C} \rightarrow M$ is equivalent to the following:

For each $\alpha \in \mathcal{C}$ of $\deg n$, choose an object X_{α} and a factorization

$$\underline{L_{\alpha}X} \rightarrow X_{\alpha} \rightarrow \underline{M_{\alpha}X}$$

of the natural map.



- This can be done independently for each object of $\deg n$

§ 5. Maps between Reedy diagrams

Same situation as above: $\mathcal{C}, M, \alpha \in \mathcal{C}, \deg \alpha = n$

- $X, Y: \mathcal{C} \rightarrow M$ diagrams

Goal: Define $X \Rightarrow Y$ inductively on filtrations.

Recall: $F^0\mathcal{C}$ contains no non-identity maps.

$\leadsto f: X|_{F^0\mathcal{C}} \Rightarrow Y|_{F^0\mathcal{C}}$ is completely determined by choosing a map $X_{\alpha} \rightarrow Y_{\alpha}$ for every object $\alpha \in \mathcal{C}$ with $\deg \alpha = 0$.

Suppose $f: X|_{F^{n-1}\mathcal{C}} \Rightarrow Y|_{F^{n-1}\mathcal{C}}$

For every $\alpha \in \mathcal{C}$ with $\deg \alpha = n$, we have:

$$\begin{array}{ccccc}
 \operatorname{colim}_{(i^n \downarrow \alpha)} X & \longrightarrow & X_\alpha & \longrightarrow & \lim_{(\alpha \downarrow i^n)} X \\
 \downarrow & & & & \downarrow \\
 \operatorname{colim}_{(i^n \downarrow \alpha)} Y & \longrightarrow & Y_\alpha & \longrightarrow & \lim_{(\alpha \downarrow i^n)} Y
 \end{array}$$

Proposition (*)



$$\begin{array}{ccccc}
 L_\alpha X & \longrightarrow & X_\alpha & \longrightarrow & M_\alpha X \\
 \downarrow & & \downarrow & & \downarrow \\
 L_\alpha Y & \longrightarrow & Y_\alpha & \longrightarrow & M_\alpha Y
 \end{array}$$

$\left\{ \text{Extensions of } f \text{ to } F^n C \right\} \xleftrightarrow{\text{correspondence}} \left\{ \text{Choice, for each } \alpha \in C \text{ with } \deg n, \text{ of a } \right.$
 $\left. \text{dashed arrow that makes squares commute} \right\}$

Suppose $A, B, X, Y: C \rightarrow M$ are Reedy diagrams and the following commutes:

(1)

$$\begin{array}{ccc}
 A & \longrightarrow & X \\
 \downarrow & \nearrow h & \downarrow \\
 B & \longrightarrow & Y
 \end{array}$$

where h is only defined on $\underline{B|_{F^{n-1}C}}$.

Then, for every $\alpha \in C$ with $\deg n$, we have an induced diagram:

(2)

$$\begin{array}{ccc}
 A_\alpha & \xrightarrow{L_\alpha B} & X_\alpha \\
 \downarrow L_\alpha A & & \downarrow \\
 B_\alpha & \xrightarrow{\quad} & Y_\alpha \times_{M_\alpha Y} M_\alpha X
 \end{array}$$

$\nearrow h_\alpha$ (dashed blue arrow)

{ h can be extended to $F^n C$ so that (1) commutes }

\updownarrow correspondence

{ for every $\alpha \in C$ of $\deg n$, there is a map $B_\alpha \xrightarrow{h_\alpha} X_\alpha$ so that (2) commutes }

• Have the ability to define the Reedy model structure.

\Rightarrow statement and proof next week featuring Sayantan!