11.03.2021

The projective model structure on chain complexes (and some other model categories)

Sources: • "Homotopy theories and model categories" (Dwyer + Spalinski)
• "Review of model categories" (G.I. Dungan)

Goal: To describe a model structure on (bounded below) chain complexes of R-modules.

§ 1. Preliminarisms and statement of the model structure

Notation / definitions:

- R : a ring with unity
- Mod_R : category of left R-modules.
- Ch_R : category of (non-negatively) graded chain complexes of R-modules.
  - objects: collections of R-modules together with boundary maps ∂_n:

$$M_\bullet := \{M_n\}_{n \geq 0}$$

$\forall\, n \geq 1 \rightsquigarrow \partial_n : M_n \longrightarrow M_{n-1}$ such that

$$\partial_n \circ \partial_{n+1} = 0$$

(ie) $\mathrm{Im}(\partial_{n+1}) \subseteq \ker(\partial_n)$

- $\ker(\partial_n) =$ "n-cycles" $=: Z_n(M_\bullet)$
- $\mathrm{Im}(\partial_{n+1}) =$ "n-boundaries" $=: B_n(M_\bullet)$

  - morphisms: $f : M_\bullet \to N_\bullet$ such that f is a collection of R-module homomorphisms indexed by $\mathbb{Z}_{\geq 0}$ s.t.

$f = \{ f_n : M_n \to N_n \mid n \geq 0 \}$ respects the boundary maps

$$\begin{tikzcd}
\cdots \arrow[r] & M_{n+1} \arrow[r] \arrow[d, "f_{n+1}"'] & M_n \arrow[r, "\partial^M_n"] \arrow[d, "f_n"'] & M_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r] & M_0 \arrow[d, "f_0"'] \\
\cdots \arrow[r] & N_{n+1} \arrow[r] & N_n \arrow[r, "\partial^N_n"'] & N_{n-1} \arrow[r] & \cdots \arrow[r] & N_0
\end{tikzcd}$$

(ie) $f_{n-1}\partial_n^M = \partial_n^N f_n \qquad \forall \quad n \geq 1$

- Homology functor:
  - For $n \geq 1$: $H_n(M_\bullet) := \text{Ker}(\partial_n) / \text{Im}(\partial_{n+1})$
  - $H_0(M_\bullet) := M_0 / \text{Im}(\partial_1)$

⇝ $H_n : Ch_R \longrightarrow Mod_R$ is a functor

$\boxed{\text{Def/Prop}}$ An R-module P is PROJECTIVE iff any of the following hold:

① P is a direct summand of a free R-module ;
$\left[P \oplus K \cong \bigoplus_{i \in I} R \right]$

② Every epimorphism $f : A \twoheadrightarrow P$ has a right inverse

$$\begin{tikzcd} 0 \arrow[r] & \text{Ker} f \arrow[r, hook] & A \arrow[r, two heads, "f"] & P \arrow[r] \arrow[l, bend left, dotted, "g"] & 0 \end{tikzcd}$$

"split exact" $\qquad f \circ g = 1$

③ For every epimorphism $f : A \twoheadrightarrow B$, the induced map

$$\begin{tikzcd} Mod_R(P, A) \arrow[r] & Mod_R(P, B) \\ g \arrow[r, mapsto, "f_*"'] & f \circ g \end{tikzcd}$$

is also an epi.

$$\begin{tikzcd} P \arrow[r, "g"'] & A \arrow[r, two heads, "f"'] & B \end{tikzcd}$$

↑ post-composition

- Since $Mod_R(P, -)$ is always left exact, ③ is equivalent to saying it is actually EXACT iff P is projective.

Another way to view ③: For any $h: P \to B$, $\exists$ $g: P \to A$ s.t.

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ & P \arrow[u, "h"'] \arrow[ul, dotted, "g"] \end{tikzcd}$$

$f \circ g = h$.

Theorem: The projective model structure on chain complexes

The following describes a model category structure on Ch_R:

- $W = \{$quasi-isomorphisms$\}$
- $C = \{$degree-wise monos with projective cokernels$\}$
- $F = \{$degree-wise epis in $\underbrace{\text{nonzero}}_{n \geq 1}$ degrees$\}$

- fibrant objects: $M_\bullet$ s.t. $M_\bullet \to 0$ is a fibration
  $\Rightarrow$ so every chain complex $M_\bullet$
- cofibrant objects: $M_\bullet$ s.t. $0 \to C_\bullet$ is a cofibration $\Rightarrow$ chain complexes with projective R-modules in every degree.

What are cokernels? For $f: X \to Y$, $\operatorname{coker} f = \operatorname{coeq}(f, 0_{XY})$

(ie) colimit of:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & Y \arrow[r] \arrow[dr] & Q \arrow[d, dashed] \\ & & Q' \end{tikzcd}$$

(ie) $\operatorname{coker}(f) = \{\text{object } Q\} \cup \{\text{morphism } q: Y \to Q\}$ s.t. $q \circ f = 0$, (ie)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "0_{XQ}"'] & Y \arrow[d, "q"] \\ & Q \end{tikzcd}$$

And, any other such $q': Y \to Q'$ can be obtained by composing q with a unique morphism $u: Q \to Q'$:

$$\begin{tikzcd} & Y \arrow[d, "q"] \arrow[ddr, bend left, "q'"] & \\ X \arrow[ur, "f"] \arrow[r, "0"] \arrow[drr, bend right, "0"'] & Q \arrow[dr, dashed, "\exists! u"] & \\ & & Q' \end{tikzcd}$$

(q' factors uniquely through q.)

§ 2. Proof of the model structure

Straightforward properties:

- W, C, F contain all identity maps.
  - They are also closed under composition.
    - W - defined using a functor
    - C - If f, g have projective cokernels, so does fg
- Mod_R has all small (co)limits
  - (co)limits in Ch_R are computed degree-wise.

$\Rightarrow$ Ch_R is (co)complete.

- W satisfies 2-out-of-3: $H_n(-)$ is a functor.

$$(g\circ f)_* = g_*\circ f_* \qquad\qquad (gf)_*^{-1} = f_*^{-1} g_*^{-1}$$

If $f, gf \in W \rightsquigarrow (gf)_* (f_*^{-1} g_*^{-1}) = g_* f_* f_*^{-1} g_*^{-1}$

$$\rightsquigarrow g_*^{-1} = (gf)_*^{-1} f_*$$

Retracts:

Suppose $f, g \in \mathrm{Mor}(\mathrm{Ch}_R)$ is such that $\exists\ i, r, i', r'$ s.t.:

$$\begin{tikzcd}
M_\bullet \arrow[r, "i"] \arrow[d, "f"'] \arrow[rr, bend left, dashed, "1"] & A_\bullet \arrow[r, "r"] \arrow[d, "g"] & M_\bullet \arrow[d, "f"] \\
N_\bullet \arrow[r, "i'"'] \arrow[rr, bend right, dashed, "1"'] & B_\bullet \arrow[r, "r'"'] & N_\bullet
\end{tikzcd}$$

ie "f is a retract of g"

- If $g\in W$ ... then g_* is an isomorphism in homology. So, pass the above diagram to k^{th} homology (can do this by functoriality) for some $k \geq 0$.

$$\begin{tikzcd}
H_k(M_\bullet) \arrow[r, "i_*"] \arrow[d, "f_*"'] \arrow[rr, bend left, dashed, "1"] & H_k(A_\bullet) \arrow[r, "r_*"] \arrow[d, "g_*"'] & H_k(M_\bullet) \arrow[d, "f_*"] \\
H_k(N_\bullet) \arrow[r, "i'_*"'] \arrow[rr, bend right, dashed, "1"'] & H_k(B_\bullet) \arrow[r, "r'_*"'] \arrow[u, bend right, dashed, "g_*^{-1}"'] & H_k(N_\bullet)
\end{tikzcd}$$

Check: $f_*^{-1} := r_* g_*^{-1} i'_*$ works.

Remember that g_* must be an isomorphism $(g \in \mathcal{W})$, so its inverse is well-defined.

- $f_*(r_* g_*^{-1} i'_*) = r'_* g_* g_*^{-1} i'_*$ by the right square
$= r'_* \, 1 \, i'_*$
$= 1$

- $(r_* g_*^{-1} i'_*) f_* = r_* g_*^{-1} g_* i_*$ by the left square
$= r_* \, 1 \, i_*$
$= 1$

$\Rightarrow f \in \mathcal{W}$.

- If $g \in \mathcal{C}$ ... then g is a degree-wise mono with projective cokernels in each degree.

Let $k \geq 0$.

- **f_k is monic:**

To show that f_k is monic: suffices to show that for all R-modules Q,

$$(f_k)_* : \mathrm{Mod}_R(Q, M_k) \longrightarrow \mathrm{Mod}_R(Q, N_k)$$
$$h \longmapsto f_k h$$

is INJECTIVE.

ⓘⓔ $f_k h = 0 \Rightarrow h = 0$

Suppose $f_k h = 0$. Then,

$$\begin{tikzcd} & Q \arrow[dl] \arrow[d, "h"] & & \\ {} & M \arrow[r, "i_k"] & {} \arrow[r, "r_k"] & M \end{tikzcd}$$

$$\begin{tikzcd}
0 & M_K \arrow[r] \arrow[d, "f_K"'] & A_K \arrow[r] \arrow[d, "g_K"'] & M_K \arrow[d, "f_K"'] \\
& N_K \arrow[lu] \arrow[r, "i'_K"'] & B_K \arrow[r, "r'_K"'] & \underset{7}{N_K}
\end{tikzcd}$$

$$\rightsquigarrow \quad g_K(i_K h) = i'_K f_K h$$

$$= i'_K(0)$$

$$= 0$$

g_K monic $\Rightarrow$ $i_K h = 0$

Now, $h = id_{M_K} h$

$$= (r_K i_K) h$$

$$= r_K(i_K h)$$

$$= r_K(0)$$

$\therefore h = 0$

$\Rightarrow$ f_K is monic.

ⓔ $\operatorname{coker}(f) = \{\text{object } Q\} \cup \{\text{morphism } g: Y \to Q\}$
s.t. $g \circ f = 0$, ⓔ

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rd, "0_{XQ}"'] & Y \arrow[d, "g"] \\
& Q
\end{tikzcd}$$

And, any other such $g': Y \to Q'$ can be obtained by composing g with a unique morphism $u: Q \to Q'$:

$$\begin{tikzcd}
& Y \arrow[d, "g"] \arrow[rd, bend left, "g'"] & \\
X \arrow[ru, "f"] \arrow[r, "0"] & {} & {}
\end{tikzcd}$$

$$\begin{tikzcd}
Q \arrow[rd, dashed, "\exists ! u"] & \\
\arrow[r, bend right, "0"'] & Q'
\end{tikzcd}$$

(q' factors uniquely through q.)

- coker(f_k) is projective:

As coker(f_k) = coeq$(f_k, 0_{M_k N_k})$, we have the following: (by colimits!)

$$\begin{tikzcd}
M_k \arrow[r, "i_k"] \arrow[d, "f_k"'] \arrow[dd, bend right=60, "0"'] & A_k \arrow[r, "r_k"] \arrow[d, "g_k"] \arrow[rd, bend left, "0"] & M_k \arrow[d, "f_k"] \arrow[dd, bend left=60, "0"] \\
N_k \arrow[r, "i'_k"'] \arrow[d, "q_f"'] & B \arrow[r, "r'_k"'] \arrow[d, "q_g"'] & N_k \arrow[d, "q_f"] \arrow[ld, bend left] \\
\operatorname{coker}(f_k) \arrow[r, "u"'] & \operatorname{coker}(g_k) \arrow[r, "v"'] & \operatorname{coker}(f_k)
\end{tikzcd}$$

unique!

To show coker(f_k) is projective: use characterization:

"For every epi $\varphi : C \twoheadrightarrow D$, the induced map

$$\begin{aligned}
\mathrm{Mod}_R(\operatorname{coker} f_k, C) &\longrightarrow \mathrm{Mod}_R(\operatorname{coker} f_k, D) \\
g &\xmapsto[\varphi_*]{} \varphi \circ g
\end{aligned}$$

is also an epi"

(maps out of coker f are all induced)

post-composition

(PUSH OUT)

Let $\varphi : C \twoheadrightarrow D$ be an epi and let $h \in \mathrm{Mod}_R(\operatorname{coker} f_k, D)$

Then,

$$\begin{tikzcd}
M_k \arrow[r, "i_k"] \arrow[d] & A_k \arrow[r, "r_k"] \arrow[d] & M_k \arrow[d] \\
{} & {} & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[d, "f_k"] \arrow[dd, bend right=60, "0"'] & {} \arrow[d, "g_k"] \arrow[dd, bend left=40, "0"] & {} \arrow[d, "f_k"] \arrow[dd, bend left=60, "0"] \\
N_k \arrow[r, "i'_k"'] \arrow[d, "q_f"'] & B \arrow[r, "r'_k"'] \arrow[d, "q_g"'] & N_k \arrow[d, "q_f"'] \\
\operatorname{coker}(f_k) \arrow[r, "u"'] & \operatorname{coker}(g_k) \arrow[r, "v"'] \arrow[d, "m"'] & \operatorname{coker}(f_k) \arrow[d, "h"] \\
& C \arrow[r, two heads, "\varphi"'] & D
\end{tikzcd}$$

As $hv \in \mathrm{Mod}_R(\operatorname{coker}(g_k), D)$ and $\operatorname{coker}(g_k)$ is projective, then $\exists$

$m: \operatorname{coker}(g_k) \longrightarrow C$ s.t.

$$\varphi m = hv$$

Now, by the second and third rows:

$$r'_k i'_k = 1$$

$$vu = 1$$

Thus,

$$hvu = h$$

Let $\ell := mu$ and notice that

$$\varphi \ell = \varphi(mu) = hvu = h$$

So $\ell : \operatorname{Coker}(f_k) \to C$ is s.t. $\varphi_*(\ell) = h$

$\Rightarrow \varphi_*$ is an epi

$\Rightarrow \operatorname{coker}(f_k)$ is projective.

$\Rightarrow f \in \mathcal{C}$ so $\mathcal{C}$ is closed under retracts.

- If $g \in \mathcal{F}$ ... a (now) simple diagram chase.
  [similar to showing degree-wise monos in case of $\mathcal{C}$]

$\Rightarrow \mathcal{W}, \mathcal{F}$, and $\mathcal{C}$ are closed under retracts. $\square$

Lifting: (Prove only some parts)

(1) $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$

Suppose

$$\begin{tikzcd} A_\bullet \arrow[r, "g"] \arrow[d, hook, "i"'] & C_\bullet \arrow[d, two heads, "p", "\sim"'] \\ B_\bullet \arrow[r, "h"'] & D_\bullet \end{tikzcd}$$

epi in all nonzero degrees.

Claim 1: p_0 is also an epimorphism.

[Use five lemma with two exact rows given by cokernel of ∂_1

and use the fact that $(p_0)_*$ is an isomorphism ⌟

Claim 2: $\ker p_\bullet$ is an acyclic chain complex $\left(H_n(\ker p) = 0 \ \forall n \right)$

⌜ WHY?

As p is an epi in every degree, we get a SES in Ch_R:

$$\begin{tikzcd} 0 \arrow[r] & \ker p \arrow[r, hook] & C_\bullet \arrow[r, two heads, "p"] & D_\bullet \arrow[r] & 0 \end{tikzcd}$$

So, we get a long exact sequence in homology ...

$$\begin{tikzcd} \cdots \arrow[r] & H_n(\ker p) \arrow[r] & H_n(C_\bullet) \arrow[r, "\cong"] & H_n(D_\bullet) \arrow[dll] \\ & H_{n-1}(\ker p) \arrow[r] & H_{n-1}(C_\bullet) \arrow[r, "\cong"] & H_{n-1}(D_\bullet) \arrow[dlll] \\ \cdots \arrow[r] & H_0(\ker p) \arrow[r] & H_0(C_\bullet) \arrow[r, "\cong"] & H_0(D_\bullet) \end{tikzcd}$$

As p_* is an iso.

$\Rightarrow$ $H_n(D_\bullet) \xrightarrow{\partial} H_{n-1}(\ker p)$ is s.t.

$\ker \partial = H_n(D_\bullet)$

$\Rightarrow$ $\partial = 0$

$\Rightarrow$ $H_{n-1}(\ker p) = 0$ ⌟

To prove $\exists$ a lift $f : B_\bullet \to C_\bullet$, we use an inductive method.

$$\begin{tikzcd} A_\bullet \arrow[r, "g"] \arrow[d, hook, "i"'] & C_\bullet \arrow[d, two heads, "p", "\sim"'] \\ B_\bullet \arrow[r, "h"'] \arrow[ur, dashed] & D_\bullet \end{tikzcd}$$

Define f_0 first.

As i_0 is a cofibration $\Rightarrow$ $P_0 := \operatorname{coker}(i_0)$ is projective

That is, $P_0 = B_0 / \operatorname{Im}(i) \cong B_0 / A_0$ is projective

And, $B_0 \cong B_0/A_0 \oplus A_0$

$$\Rightarrow B_0 \cong P_0 \oplus A_0$$

$$\begin{tikzcd}
A_0 \arrow[r, "g_0"] \arrow[d, hook, "i_0"'] & C_0 \arrow[d, two heads, "p_0", "\sim"'] \\
P_0 \oplus A_0 \arrow[r, "h_0"'] \arrow[ur, dashed, "f_0 = l_0 \oplus g_0"] & D_0
\end{tikzcd}$$

Since $p_0 : C_0 \twoheadrightarrow D_0$ is an epi, $\exists$ $l_0 : P_0 \to C_0$ s.t.

$$p_0 \circ l_0 = h_0|_{P_0}$$

Define $f_0 := l_0 \oplus g_0$

Inductive hypothesis: For $0 < k < n$, assume f_k satisfies:

① $\partial f_k = f_{k-1} \partial$ } f is a chain map

② $p_k f_k = h_k$

③ $f_k i_k = g_k$ } 2 triangles in lift diagram commute.

$$\begin{tikzcd}
A_n \arrow[r, "g_n"] \arrow[d, hook, "i_n"'] & C_n \arrow[d, two heads, "p_n", "\sim"'] \\
B_n \arrow[r, "h_n"'] \arrow[ur, dashed, "\tilde{f}_n"] & D_n
\end{tikzcd}$$

In the same way as f_0, we construct $\tilde{f}_n : B_n \to C_n$.

$\rightsquigarrow$ ② and ③ hold by the lift.

BUT ... haven't guaranteed that ① holds (nothing regarding the boundary maps was used in the construction)

To ensure it does hold, we define a "difference map" that measures the "failure of $\tilde{f}_n$ to satisfy ①", call this ε, and then REMOVE its contributions.

Let $\varepsilon : B_n \longrightarrow C_{n-1}$ be defined by:

$$\varepsilon := \partial^C_n \tilde{f}_n - f_{n-1}\partial^B_n$$

(n-1)-cycles of $\ker p_\bullet$

$$Z_{k-1}(M) = \begin{cases} M_0 & \text{if } k = 1 \\ \ker(\partial_{k-1} : M_{k-1} \to M_{k-2}) \end{cases}$$

<u>Claim</u>: ε induces a map $\varepsilon' : P_n \longrightarrow Z_{n-1}(\ker p)$.

⌈ proof uses the definition of ε and:

(1) $\partial\varepsilon = 0$ as f_{n-1} satisfies ①

(2) $p_{n-1}\varepsilon = 0$ as $p_n \tilde{f}_n = h_n$ commutes with ∂

(3) $\varepsilon i_n = 0$ since $\tilde{f}_n i_n = g_n$ commutes with ∂.

And we construct lifts using kernels as equalizers and cokernels as coequalizers. ⌋

Using this, let $j_n : \ker p_n \hookrightarrow C_n$ and $\pi_n : B_n \twoheadrightarrow P_n$.

$$\varepsilon'' := j_n \varepsilon' \pi_n : B_n \longrightarrow C_n$$

and

$$f_n := \tilde{f}_n - \varepsilon''$$

⌈ Some reasoning why f_n satisfies ①,②,③:

- $\varepsilon' : P_n \to \ker p_n$, it doesn't affect ②
- As $i_n : A_n \hookrightarrow A_n \oplus P_n$, ε' doesn't affect ③
- $\partial f_n = f_{n-1}\partial$ by choices we made for lifts, and ε, ε' defns. ⌋

Therefore,

$$\begin{tikzcd} A_\bullet \arrow[r, "g"] \arrow[d, hook, "i"'] & C_\bullet \arrow[d, two heads, "p", "\sim"'] \\ B_\bullet \arrow[r, "h"'] \arrow[ur, "\exists f"] & D_\bullet \end{tikzcd}$$

$\Rightarrow$ $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ has lifting.

(2) $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$

First, we need some definitions and lemmas.

Define two chain complexes associated to a ring R : (actually, we can do this for any R-module M).

$\boxed{\text{Def}}$ Let $n \in \mathbb{N}$, $n \geq 1$.

The n-DISK CHAIN COMPLEX of R is given by R-modules $\{D^n(R)\}_k$ as follows:

$$D^n(R)_k := \begin{cases} 0 & \text{if } k \neq n, n-1 \\ R & \text{else} \end{cases}$$

And $\partial_n = id$, $\partial_k = 0$ $\forall$ $k \neq n$

$$\to 0 \to 0 \cdots \to \underset{\substack{\uparrow \\ n^{th} \text{ degree}}}{R} \xrightarrow{\partial_n} R \to 0 \to 0 \to \cdots\cdots \to 0$$

$\boxed{\text{Def}}$ Let $n \geq 0$.

The n-SPHERE CHAIN COMPLEX of R is defined by:

$$S^n(R)_k := \begin{cases} 0 & \text{if } k \neq n \\ R & \text{if } k = n \end{cases}$$

$$\cdots\ 0 \to 0 \to \underset{\substack{\uparrow \\ n^{th} \text{ degree}}}{R} \xrightarrow{0} 0 \xrightarrow{0} 0 \ \cdots \quad \cdots\cdots \quad \cdots \longrightarrow 0$$

$\boxed{\text{Lemma 1}}$ Let $M_\bullet \in Ch(R)$. Then,

$$Ch_R(D_n(R), M_\bullet) \xrightarrow{\cong} Mod_R(R, M_n)$$

$$\underset{\nwarrow \text{ map of chain complexes}}{f} \longmapsto \underset{\uparrow \text{ map of degree } n.}{f_n}$$

This is an isomorphism.

⌈ Almost immediate

$\varphi_n : R \to M_n$ $\qquad$ $\varphi_{n-1} : R \to M_{n-1}$

If $f_n = g_n$

Then

$$\begin{tikzcd} R \arrow[r, "id"] \arrow[d, "g_n = f_n"'] & R \arrow[d, "f_{n-1}"] \\ M_n \arrow[r, "\partial"'] & M_{n-1} \end{tikzcd}$$

$f_{n-1}\, id = \partial f_n$

$g_{n-1}\, id = \partial g_n$

$f_{n-1} = g_{n-1}$. ⌋

Actually ... $D_n(-) \dashv \left(Ch_R \to Mod_R \right)$

$$M_\bullet \mapsto M_n$$

$\boxed{\text{Corollary}}$ If A is a projective R-module, then for any epimorphism $p: M_\bullet \twoheadrightarrow N_\bullet$,

$$\mathrm{Ch}_R(D^n(A), M_\bullet) \cong \mathrm{Mod}_R(A, M_n) \longrightarrow \mathrm{Mod}_R(A, N_n) \cong \mathrm{Ch}_R(D^n(A), N_\bullet)$$

is also an epimorphism.

(ie) A is a "projective chain complex" in that for any epimorphism of chain complexes $p: M \twoheadrightarrow N_\bullet$, any map $D_n(A) \to N$ LIFTS over p to a map $D_n(A) \to M$

$$\begin{tikzcd}
 & M \arrow[d, two heads, "p"] \\
D^n(A) \arrow[ur, dashed] \arrow[r] & N
\end{tikzcd}$$

$\boxed{\text{Lemma 2}}$ Suppose $P_\bullet \in \mathrm{Ch}(R)$ is acyclic (all homology groups are trivial) with each P_n a projective R-module.

Then, each module $Z_n(P_\bullet)$ is also projective and

$$P_\bullet \underset{\substack{\uparrow \\ \text{iso as a chain complex.}}}{\cong} \bigoplus_{n\geq 1} D^n(Z_{n-1}(P_\bullet))$$

Back to lifting: Suppose

$$\begin{tikzcd}
A_\bullet \arrow[r, "g"] \arrow[d, hook, "i"', "\sim"] & C_\bullet \arrow[d, two heads, "p"] \\
B_\bullet \arrow[r, "h"'] & D_\bullet
\end{tikzcd}$$

Let $P_\bullet = \mathrm{coker}(i)$. (cokernel in every degree)

SES $\quad 0 \to A_\bullet \to B_\bullet \to P_\bullet \to 0$

$\rightsquigarrow$ long exact sequence in homology and $H_n(A_\bullet) \cong H_n(B_\bullet)$

$H_n(P_\bullet) = 0 \ \forall n.$

$\Rightarrow$ $P_\bullet$ is acyclic and each P_n is projective, we're in the situation of Lemma 2

$\rightsquigarrow$ $Z_n(P_\bullet)$ is projective too and $P_\bullet \cong \bigoplus_{n\geq 1} D^n(Z_{n-1}(P_\bullet))$ and $D^n(Z_{n-1}(P_\bullet))$ is projective in each degree (direct summand of a projective module)

As $B_\bullet \cong A_\bullet \oplus P_\bullet$ and p is an epi, $\exists$ a lift $\ell : P_\bullet \to C_\bullet$ s.t.

$$\begin{tikzcd}
 & & C_\bullet \arrow[d, "p"] \\
P_\bullet \arrow[r, "\text{inclusion}"'] \arrow[rru, dashed, "\ell"] & A_\bullet \oplus P_\bullet \arrow[r, "h"'] & D_\bullet
\end{tikzcd}$$

commutes (by corollary to Lemma 1)!

$\Rightarrow$ Take $g \oplus \ell$, and this is our desired lift

$$\begin{tikzcd}
A_\bullet \arrow[r, "g"] \arrow[d, hook, "i"'] & C_\bullet \arrow[d, "p"] \\
B_\bullet \arrow[r, "h"'] \arrow[ru, dashed, "\sim\ \exists g\oplus\ell"] & D_\bullet
\end{tikzcd}$$

□

Factorization:

- Want to use SOA.
- Need sequentially small domains.

Def An object $M_\bullet \in Ch_R$ is SEQUENTIALLY SMALL iff:

(1) Only finitely many $M_k \neq 0$; and

(2) Each M_k has a finite presentation

↳ finitely generated R-module with some epi $\varphi : F \twoheadrightarrow M_k$ where F is free and $\ker\varphi$ is fin. generated $\Rightarrow F/\ker\varphi \cong M$

⇝ (e) isomorphic to cokernel of a map between 2 finitely generated free R-modules.

The Gluing Construction and the SOA

morphisms between chain complexes.

Let $F := \{f_i : A_i \to B_i\}_{i \in I} \subseteq Mor(Ch_R)$

Suppose that $p : M_\bullet \to N_\bullet \in Mor(Ch_R)$.

We want to factor f as a composite

$$\begin{tikzcd} M_\bullet \arrow[r, "r"] \arrow[rr, bend right, "p"'] & Q_\bullet \arrow[r, "e"] & N_\bullet \end{tikzcd}$$

s.t. $e : Q_\bullet \to N_\bullet$ has the RIGHT LIFTING PROPERTY with respect to all the maps in F.

fib ⟺ RLP wrt $0 \to D^n$ $\forall n \geq 1$

↳ acyclic fib ⟺ RLP wrt $j_n : S^{n-1} \to D^n$ $\forall n \geq 0$.

$F = \{j_n\}_{n \geq 0}$

— $Q_\bullet = N_\bullet$ — ok, but we want the factorization s.t. $Q_\bullet$ is as close to $M_\bullet$ as possible

So, we use the following construction. (the gluing construction!)

- For each $i \in I$, define ← pairs of maps in $Ch(R)$.

$$S(i) := \left\{ (g,h) \;\middle|\; g: A_i \to M_0,\ h: B_i \to N_0 \text{ s.t. } (*) \text{ commutes} \right\}$$

(*)

$$\begin{tikzcd}
A_i \arrow[r, "g"] \arrow[d, "f_i"'] & M_0 \arrow[d, "p"] \\
B_i \arrow[r, "h"'] & N_0
\end{tikzcd}$$

$$\begin{tikzcd}
\underset{B_i}{\circ} & \underset{A_i}{\circ} \arrow[l] \arrow[r] & \underset{M_0}{\circ}
\end{tikzcd}$$

- Define the gluing construction $G^1(\mathcal{F}, p) \in Ob(Ch_R)$ given by the following pushout diagram:

"gluing a copy of B_i to M_0 $\forall$ such diagrams"

$$\begin{tikzcd}
\coprod_{i\in I} \coprod_{(g,h)\in S(i)} A_i \arrow[r, "+_i +_{(g,h)} g"] \arrow[d, "\coprod_{i\in I} f_i"] & M_0 \arrow[r, "p"] \arrow[d, "i_1"] & N_0 \\
\coprod_{i\in I} \coprod_{(g,h)\in S(i)} B_i \arrow[r, "+_i +_{(g,h)} h"'] & G^1(\mathcal{F}, p) \arrow[ur, dashed, "p_1"'] &
\end{tikzcd}$$

In words: This is similar to the singular complex construction, in that we are gluing a copy of B_i to M_0 along A_i FOR EVERY COMMUTATIVE DIAGRAM of the form (*)

Then, there is a natural map

$$i_1: M_0 \to G^1(\mathcal{F}, p)$$

In a category with coprods, pushout = coequalizer = colimit

By universal property of colimits, the commutative diagrams given in (*) induce a map

$$p_1 : G^1(\mathcal{F}, p) \to N_0 \quad \text{s.t.}$$

$$p_1 i_1 = p$$

colimit of diagram $F : D \to C$ is a cone & from diagram to colim F s.t. for any other cone $p : D \to N_0$, ∃! $p_\infty : \text{colim} \to N_0$ so that $p = p_\infty i_\infty$

↑ object

Now, repeat + proceed by induction:

- For $k \geq 1$, define $G^k(\mathcal{F}, p)$ and maps $G^k(\mathcal{F}, p) \to Y$ by:

$$G^k(\mathcal{F}, p) = G^1(\mathcal{F}, p_{k-1})$$

and

$$p_k = (p_{k-1})_1$$

↖ taking the map that arises from colimit!

(ie), we have

$$\begin{tikzcd}
 & & G^\infty(\mathcal{F},p) \arrow[dll, bend right=60, "p_\infty"'] & & & \\
M_0 \arrow[r, "i_1"] \arrow[urr, "i_\infty"] \arrow[d, "p"] & G^1(\mathcal{F},p) \arrow[r] \arrow[ur] \arrow[d, "p_1"] & G^2(\mathcal{F},p) \arrow[r, "i_3"] \arrow[u] \arrow[d, "p_2"] & \cdots \arrow[r, "i_k"] \arrow[ul] & G^k(\mathcal{F},p) \arrow[r] \arrow[d, "p_k"] & \cdots \\
N_0 \arrow[r, equal] & N_0 \arrow[r, equal] & N_0 \arrow[r, equal] & \cdots \arrow[r, equal] & N_0 \arrow[r, equal] & \cdots
\end{tikzcd}$$

∃! p_∞

as this is another cone from diagram to N_0

Functor $X : Z^{t} \to C$

$M_0 \to G^1(\mathcal{F},p) \to \cdots \to G^k(\mathcal{F},p) \to \cdots$

↑ is a sequential direct system ↑

colimit of this is SEQUENTIAL COLIMIT OF THE objects $X(n)$.

$\boxed{\text{Def}}$ The infinite gluing construction, denoted $G^\infty(\mathcal{F}, p)$, is the sequential colimit of the top row in the green diagram above:

- By colimit ⇝ there are natural maps

$$i_\infty : M_0 \to G^\infty(\mathcal{F}, p) \quad \text{and}$$

$$p_\infty : G^\infty(\mathcal{F}, p) \longrightarrow N_0 \quad \text{s.t.} \quad p_\infty i_\infty = p$$

$\boxed{\text{Proposition}}$ With the setup above, suppose further that $A_i \in Ch_R$ is sequentially
[SOA] small (only finitely many degrees nonzero, and each module has a finite presentation).

Then,

$P_\infty : G^\infty(\mathcal{F}, p) \longrightarrow N_\cdot$ has the right lifting property wrt all maps in $\mathcal{F}$.

Now, we can prove factorization with the help of the following lemma:

+ SOA.

To use the gluing construction, we want a set of maps from sequentially small chain complexes ⇝ we already know 2 simple ones: $D^n(R)$ and $S^n(R)$!

$\boxed{\text{Lemma } 3}$ The map $q : Q_\cdot \to N_\cdot$ is :

① a FIBRATION iff q has the RLP with respect to the maps

$$\{0 \to D^n(R)\} \ \forall n > 0.$$

② an ACYCLIC FIBRATION iff q has the RLP wrt the maps

$$\{ j_n : S^{n-1}(R) \to D^n(R)\} \ \forall n \geq 0$$

$S^{-1} = 0.$

$$\begin{tikzcd} & \uparrow & & & \\ 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & R \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

$\uparrow$ $n-1$ degree

$\boxed{\text{Proof}}$

① By Lemma 1, applied to the R-module $A = R$,

$$\mathrm{Hom}_{Ch_R}(D^n(R), N_\cdot) \xrightarrow{\cong} \mathrm{Hom}_{Mod_R}(R, N_n)$$

$$\mathrm{Hom}_{Ch_R}(D^n(R), N_\cdot) \cong \mathrm{Hom}_{R\text{-mod}}(R, N_n) \qquad f \mapsto f(1)$$

For $n\geq 1$,

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d] & Q_\bullet \arrow[d, "p"] \\ D^n(R) \arrow[r, "f"'] \arrow[ur, dashed, "j"] & N_\bullet \end{tikzcd}$$

$\cong N_n$.

$\forall\ D^n(R) \longrightarrow N$, $\exists !$

N_n

If

such a lift $j: D^n(R) \to Q$ exists, then

$p \circ j = f$ $\Rightarrow$ p is surjective.

$j \circ 0 = 0 \Rightarrow j = 0$

[whenever h is s.t. $hf = 0$ in degree ≥ 0, then $h=0$ $\Rightarrow$ f is surjective]

② Exercise.

□

(1) $(\mathcal{C}, \mathcal{F}\cap\mathcal{W})$:

Let $f: M_\bullet \to N_\bullet$ be a map in Ch_R. [this is the map we want to factor in $(\mathcal{C}, \mathcal{F}\cap\mathcal{W})$]

Let

$$\mathcal{F} := \left\{ j_n : S^{n-1}(R) \to D^n(R) \right\}_{n\geq 0}$$

Use the factorization given by SOA:

$$\begin{tikzcd} M_\bullet \arrow[r, "i_\infty"] \arrow[rr, bend right, "f"'] & G^\infty(\mathcal{F}, f) \arrow[r, two heads, "p_\infty", "\sim"'] & N_\bullet \end{tikzcd}$$

Then, $\overset{\text{SOA}}{p_\infty}$ has the RLP wrt all maps $\{j_n\}_{n\geq 0}$

$\Rightarrow$ Lemma 3 tells us that p_∞ is an acyclic fibration

Need to check: i_∞ is a cofibration (monic in each degree with projective cokernel)

In each degree n,

$$G^{k+1}(\mathcal{F}, f) = G^k(\mathcal{F}, f) \oplus \bigoplus_{\alpha \in \Lambda} R$$

$\uparrow$ possibly large.

$$\big\downarrow$$

passing to the colimit:

$$G^\infty(\mathcal{F}, f)_n = M_n \oplus \Big(\bigoplus_{\alpha \in \beta} R \Big)$$

$\Rightarrow$ monomorphism with projective cokernel

∘ proof for $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is similar, except use

$$\mathcal{F}' := \{ 0 \to D_n \}_{n \geq 1}$$

and

$$M_\circ \xrightarrow{i_\infty} G^\infty(\mathcal{F}', f) \xrightarrow{p_\infty} N_\circ$$

$\square$

Finally ... Ch_R is a model category.
called "projective".

Another model structure: "Injective" model structure on cochain complexes

$\mathcal{C} =$ degree-wise monic for positive degree

$\mathcal{F} =$ degree-wise epis with injective kernel.

A few words about Top:

Model category structure on Top:

(1) $\mathcal{W} = \{$weak homotopy equivalences$\}$

(2) $\mathcal{F} = \{$Serre fibrations$\}$

$f: X \to Y$ s.t. for each $x \in X$, the map $f_*: \pi_n(X, x) \to \pi_n(Y, f(x))$ is a bijection of pointed sets for $n = 0$ and an iso for $n \geq 1$

↑ group iso.

(3) $\mathcal{C} = LLP(\mathcal{F} \cap \mathcal{W})$

continuous functions $f: X \to Y$ that have the RLP wrt all inclusions $(id, 0): \underset{\substack{\uparrow \\ \text{standard} \\ n \text{ disk}}}{D^n} \hookrightarrow \underbrace{D^n \times I}_{\substack{\uparrow \\ \text{cylinder object} \\ \text{of } D^n.}}$