HILBERT SPACES: ORTHONORMAL BASES

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November 17, 2017

Recall definitions from II.1:

Definition 0.1. A $\mathbb{K}$ vector space V is an *inner product space* if there exists a map ("*the inner product*") $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ that satisfies the following conditions for all $x, y, z \in V, \alpha \in \mathbb{K}$

(1) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \quad \Leftrightarrow \quad x = 0$

(2) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(4) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

- $x, y \in V$ are *orthogonal* if $\langle x, y \rangle = 0$.
- $\{x_i\}_{i \in I} \in V$ collection of vectors in V is called an *orthonormal set* iff

$$\langle x_i, x_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Theorem 1. *Every inner product space V is a normed linear space with the norm*

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Definition 0.2. A *Hilbert Space* is a complete inner product space.

1. Hilbert Spaces: Orthonormal Bases

– Want to develop idea of orthonormal set further $\Rightarrow$ To extend the finite-dimensional "basis" to complete inner product spaces.

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space.

Orthonormal Basis for $\mathcal{H}$ / Complete Orthonormal System for $\mathcal{H}$:

An orthonormal set $S \subseteq \mathcal{H}$ such that if $S \subsetneq U$ for some $U \subseteq \mathcal{H}$, then U is not an orthonormal set.

– i.e., No other orthonormal set contains S as a proper subset.

Theorem II.5

Every Hilbert space $\mathcal{H}$ has an orthonormal basis.

Proof. (Standard application of Zorn's Lemma)

Let $\mathcal{C}$ be the collection of all orthonormal sets in $\mathcal{H}$.

Define the regular partial ordering on $\mathcal{C}$ (set inclusion), i.e., $S_1 < S_2$ if $S_1 \subset S_2$.

$\mathcal{C}$ is non-empty since for any $v \in V$, $\left\{ \frac{v}{\|v\|} \right\}$ is an orthonormal set $\Rightarrow$ $\mathcal{C} \neq \emptyset$.

Let $\{S_\alpha\}_{\alpha \in A}$ be any linearly ordered subset of $\mathcal{C}$ (i.e., any totally ordered subset of $\mathcal{C}$).

Then,

$$S = \bigcup_{\alpha \in A} S_\alpha \text{ is an orthonormal set and}$$

$$S_\alpha < S \quad \forall \alpha \in A \text{ (as } S_\alpha \subset S \text{ by definition of union)}$$

Why is S orthonormal?

⌜ Suppose $x, y \in S, x \neq y$. Clearly $\langle x, x \rangle = 1 = \langle y, y \rangle$ as both x, y are elements of an orthonormal set in V. Then, $x \in S_i$ and $y \in S_j$ for some $i, j \in A$. As $\{S_\alpha\}_{\alpha \in A}$ is totally ordered, then either $S_i \subset S_j$ or $S_j \subset S_i$.

WLOG, $S_i \subset S_j$, then $x, y \in S_j$ and hence since S_j is an orthonormal set, then $\langle x, y \rangle = 0$. As $x, y \in S$ were arbitrary, then it follows that S is an orthonormal set. ⌟

Now, S is orthonormal ($S \in \mathcal{C}$) and $S_\alpha < S$ for all $\alpha \in A$. By definition, then S is an upper bound of the totally ordered subset $\{S_\alpha\}_{\alpha \in A}$.

As $\{S_\alpha\}_{\alpha \in A}$ was an arbitrary totally ordered subset of $\mathcal{C}$, then every totally ordered subset has an upper bound.

By *Zorn's Lemma*, it follows that $\mathcal{C}$ has a *maximal* element. That is, there exists some $B \in \mathcal{C}$ such that if $B \subset S$ for an orthonormal set $S \subseteq V$, then $B = S$. Hence, B is an orthonormal set that is not properly contained in any other orthonormal set. By definition, B is an orthonormal basis for $\mathcal{H}$.

$\Rightarrow$ Every Hilbert space has an orthonormal basis. $\square$

BUT... Orthonormal Basis $\neq$ Vector Space Basis

Counterexample: For each $n \in \mathbb{N}$, let $e_n \in \ell^2$ be the sequence $(e_{n,k})_{k\geq 1}$ where

$$e_{n,n} = 1 \text{ and } e_{n,k} = 0 \text{ if } n \neq k$$

So $\mathcal{E} = \{e_n \mid n \in \mathbb{N}\}$ is an orthonormal basis of ℓ^2 (since $\langle x, e_j \rangle = x_j$ for all j), but NOT a vector space basis since the sequence $\left(\frac{1}{n}\right)_{n\geq 1} \in \ell^2$ is NOT a finite linear combination of elements of $\mathcal{E}$. i.e.,

$$\text{lin}\mathcal{E} \neq \ell^2$$

Analogous to finite-dimensional vector spaces, elements of a Hilbert space can be expressed as a "linear combination" of basis elements by the following theorem.

Theorem II.6

Let $\mathcal{H}$ be a Hilbert space and $S = \{x_\alpha\}_{\alpha\in A}$ an orthonormal basis. Then, for each $y \in \mathcal{H}$,

$$y = \sum_{\alpha\in A} \langle y, x_\alpha \rangle x_\alpha \qquad (1)$$

and

$$\|y\|^2 = \sum_{\alpha\in A} |\langle y, x_\alpha \rangle|^2 \qquad (2)$$

The equality in (1) means that the sum on the right-hand side converges to y in $\mathcal{H}$ independent of order.

Conversely, if $\sum_{\alpha\in A} |c_\alpha|^2 < \infty, c_\alpha \in \mathbb{K}$, then $\sum_{\alpha\in A} c_\alpha x_\alpha$ converges to an element of $\mathcal{H}$.

Proof.

To show (1): $y = \sum_{\alpha\in A} \langle y, x_\alpha \rangle x_\alpha$, let $y \in \mathcal{H}$ be arbitrary.

By *Bessel's Inequality* (Section II.1), for any finite subset $A' \subseteq A$,

$$\sum_{\alpha\in A'} |\langle y, x_\alpha \rangle|^2 \leq \|y\|^2$$

Thus, $\langle y, x_\alpha \rangle \neq 0$ for only a countable number of α's in A.

Why is $E = \{\alpha \in A \mid \langle y, x_\alpha \rangle \neq 0\}$ countable?

⌜ For each $n \in \mathbb{N}$, define

$$E_n = \left\{ \alpha \in A \;\middle|\; |\langle y, x_\alpha \rangle| > \frac{1}{n} \right\}$$

Suppose E_n is infinite.

Then, there exists a countably infinite subset $\{\alpha_m\}_{m\in\mathbb{N}} \subset E_n$ such that $\alpha_m \neq \alpha_k$ whenever $k \neq m$. Hence, by Bessel's inequality,

$$\|y\|^2 \geq \sum_{i=1}^{m} |\langle y, x_{\alpha_i} \rangle|^2$$

Taking the limit as $m \to \infty$,

$$\|y\|^2 \geq \sum_{m\in\mathbb{N}} |\langle y, x_{\alpha_m} \rangle|^2 \geq \sum_{m\in\mathbb{N}} \frac{1}{n^2}$$

As the latter sum diverges, this implies $\|y\| = \infty$ (contradiction!). Hence, each E_n is finite.

Now,

$$E = \{\alpha \in A \mid \langle y, x_\alpha \rangle \neq 0\} = \bigcup_{n\in\mathbb{N}} E_n$$

is a countable union of finite sets $\quad\Rightarrow\quad$ E is countable.

Order these indices by $\alpha_1, \alpha_2, \ldots$ and so on.

Now, $\sum_{j=1}^{n} \left|\langle y, x_{\alpha_j} \rangle\right|^2 \leq \|y\|^2$ is monotone increasing (as it is summing positive terms) and bounded (by $\|y\|^2$), hence it converges as $n \to \infty$.

Since

$$\sum_{\alpha\in A} \|\langle y, x_\alpha \rangle x_\alpha\| = \sum_{\alpha\in A} |\langle y, x_\alpha \rangle| < \infty$$

then this sum *absolutely converges*. As $\mathcal{H}$ is a Banach space, then it follows that $\sum_{\alpha\in A} \langle y, x_\alpha \rangle x_\alpha \in \mathcal{H}$ converges in $\mathcal{H}$.

For $n \in \mathbb{N}$, let $y_n = \sum_{j=1}^{n} \langle y, x_{\alpha_j} \rangle x_{\alpha_j}$.

Then, for $n > m$,

$$\|y_n - y_m\|^2 = \left\| \sum_{j=m+1}^{n} \langle y, x_{\alpha_j} \rangle x_{\alpha_j} \right\|^2$$

$$\|y_n - y_m\|^2 = \sum_{j=m+1}^{n} \left|\langle y, x_{\alpha_j} \rangle\right|^2$$

Why?

⌜ By definition of the norm on $\mathcal{H}$,

$$\left\| \sum_{j=m+1}^{n} \langle y, x_{\alpha_j} \rangle x_{\alpha_j} \right\|^2 = \left(\sum_{j=m+1}^{n} \langle y, x_{\alpha_j} \rangle x_{\alpha_j}, \sum_{j=m+1}^{n} \langle y, x_{\alpha_j} \rangle x_{\alpha_j} \right)$$

By linearity of the inner product on $\mathcal{H}$ in the first coordinate and conjugate linearity in the second coordinate, then

$$\left(\sum_{j=m+1}^{n} \langle y, x_{\alpha_j}, y \rangle x_{\alpha_j}, \sum_{j=m+1}^{n} \langle y, x_{\alpha_j} \rangle x_{\alpha_j} \right) = \sum_{j=m+1}^{n} \sum_{j=m+1}^{n} \langle y, x_{\alpha_j} \rangle \cdot \overline{\langle y, x_{\alpha_j} \rangle} \cdot \langle x_{\alpha_j}, x_{\alpha_j} \rangle$$

As $S = \{x_\alpha\}_{\alpha \in A}$ is an orthonormal set, $\langle x_{\alpha_j}, x_{\alpha_j} \rangle = 1$ and hence

$$\left(\sum_{j=m+1}^{n} \langle y, x_{\alpha_j} \rangle x_{\alpha_j}, \sum_{j=m+1}^{n} \langle y, x_{\alpha_j} \rangle x_{\alpha_j} \right) = \sum_{j=m+1}^{n} \langle y, x_{\alpha_j} \rangle \cdot \overline{\langle y, x_{\alpha_j} \rangle}$$

$$\Rightarrow \quad \|y_n - y_m\|^2 = \sum_{j=m+1}^{n} \left| \langle y, x_{\alpha_j} \rangle \right|^2$$

Thus the statement holds ⌟

Since $\sum_{j=1}^{n} \left| \langle y, x_{\alpha_j} \rangle \right|^2$ converges as $n \to \infty$, then

$$\|y_n - y_m\|^2 = \sum_{j=m+1}^{n} \left| \langle y, x_{\alpha_j} \rangle \right|^2 \quad \Rightarrow \quad (y_n)_{n \geq 1} \in \mathcal{H} \text{ is Cauchy}$$

Why?

⌜ Let $\varepsilon > 0$ be arbitrary.

$$\|y_n - y_m\|^2 = \sum_{j=m+1}^{n} \left| \langle y, x_{\alpha_j} \rangle \right|^2$$

Let $L = \lim_{n \to \infty} \sum_{j=1}^{n} \left| \langle y, x_{\alpha_j} \rangle \right|^2$.

Let $\varepsilon' = \varepsilon^2 - |L|$. Then, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left| \sum_{j=1}^{n} \left| \langle y, x_{\alpha_j} \rangle \right|^2 - L \right| < \varepsilon'$$

Since $\|y_n - y_m\|^2 = \sum_{j=m+1}^{n} \left| \langle y, x_{\alpha_j} \rangle \right|^2$, and by the reverse triangle inequality, then

$$\Rightarrow \quad \left| \|y_n - y_m\|^2 - |L| \right| < \varepsilon'$$

$$\Rightarrow \quad \|y_n - y_m\|^2 < \varepsilon' + |L|$$

$$\|y_n - y_m\| < \sqrt{\varepsilon' + |L|}$$

$$\Rightarrow \quad \|y_n - y_m\| < \varepsilon \text{ for all } n, m \geq N$$

Thus, $(y_n)_{n \geq 1} \in \mathcal{H}$ is Cauchy ⌟.

$\mathcal{H}$ is complete $\quad\Rightarrow\quad \exists$ some $y' \in \mathcal{H}$ such that

$$(y_n)_{n\geq 1} \overset{n\to\infty}{\longrightarrow} y' \in \mathcal{H}$$

$$\Rightarrow \quad y' = \lim_{n\to\infty} \sum_{j=1}^{n} \langle y, x_{\alpha_j} \rangle x_{\alpha_j}$$

Consider $\langle y - y', x_{\alpha_t} \rangle$ for some $t \in A$ (i.e., $x_{\alpha_t} \in S$). Since $\lim_{n\to\infty} y_n = y'$, by the continuity of the inner product then

$$\langle y - y', x_{\alpha_t} \rangle = \lim_{n\to\infty} \langle y - y_n, x_{\alpha_t} \rangle$$

$$\langle y - y', x_{\alpha_t} \rangle = \lim_{n\to\infty} \left\langle y - \sum_{j=1}^{n} \langle y, x_{\alpha_j} \rangle x_{\alpha_j}, x_{\alpha_t} \right\rangle$$

By linearity of the inner product in the first coordinate, then

$$\langle y - y', x_{\alpha_t} \rangle = \lim_{n\to\infty} \langle y, x_{\alpha_t} \rangle - \left\langle \sum_{j=1}^{n} \langle y, x_{\alpha_j} \rangle x_{\alpha_j}, x_{\alpha_t} \right\rangle$$

$$\langle y - y', x_{\alpha_t} \rangle = \lim_{n\to\infty} \langle y, x_{\alpha_t} \rangle - \sum_{j=1}^{n} \langle y, x_{\alpha_j} \rangle \langle x_{\alpha_j}, x_{\alpha_t} \rangle$$

Now, since $x_{\alpha_j}, x_{\alpha_t} \in S$ belong to an orthonormal set, then $\langle x_{\alpha_j}, x_{\alpha_t} \rangle = \delta_{jt} = 1$ only when $j = t$. Hence, the only element of the sum that remains occurs when $j = t$.

$$\Rightarrow \quad \langle y - y', x_{\alpha_t} \rangle = \langle y, x_{\alpha_t} \rangle - \langle y, x_{\alpha_t} \rangle \langle x_{\alpha_t}, x_{\alpha_t} \rangle$$

As $\langle x_{\alpha_t}, x_{\alpha_t} \rangle = 1$, then

$$\langle y - y', x_{\alpha_t} \rangle = \langle y, x_{\alpha_t} \rangle - \langle y, x_{\alpha_t} \rangle$$

$$\Rightarrow \quad \langle y - y', x_{\alpha_t} \rangle = 0$$

As $x_{\alpha_t} \in S$ was arbitrary, $\langle y - y', x_\alpha \rangle = 0$ for all $x_\alpha \in S$.

Since S is an orthonormal basis, it is not contained in any larger orthonormal set, and hence it follows that $y - y' = 0$, $y = y'$.

$$y = \lim_{n\to\infty} \sum_{j=1}^{n} \langle y, x_{\alpha_j} \rangle x_{\alpha_j} = \sum_{\alpha \in A} \langle y, x_\alpha \rangle x_\alpha \qquad \Rightarrow \qquad \text{(1) holds}$$

To show (2) holds: Let $y \in \mathcal{H}$ be arbitary as before.

Recall that since $\sum_{j=1}^{N} \left|\langle y, x_{\alpha_j}\rangle\right|^2 \leq \|y\|^2$ is monotone increasing (as it is summing positive terms) and bounded (by $\|y\|^2$), it converges as $N \to \infty$, That is,

$$\sum_{\alpha \in A} |\langle y, x_\alpha\rangle|^2 < \infty$$

Remains to show that $\|y\|^2 = \sum_{\alpha \in A} |\langle y, x_\alpha\rangle|^2$.

By (1),

$$y = \lim_{n\to\infty} \sum_{j=1}^{n} \langle y, x_{\alpha_j}\rangle x_{\alpha_j}$$

By continuity of the inner product, then

$$\|y\|^2 = \lim_{n\to\infty} \left\| \sum_{j=1}^{n} \langle y, x_{\alpha_j}\rangle x_{\alpha_j} \right\|$$

By the definition of $\|\cdot\|$ on $\mathcal{H}$,

$$\|y\|^2 = \lim_{n\to\infty} \left(\sum_{j=1}^{n} \langle y, x_{\alpha_j}\rangle x_{\alpha_j}, \sum_{j=1}^{n} \langle y, x_{\alpha_j}\rangle x_{\alpha_j} \right)$$

By linearity (in 1st coordinate) and conjugate linearity (in 2nd coordinate) of the inner product,

$$\|y\|^2 = \lim_{n\to\infty} \sum_{j=1}^{n} \langle y, x_{\alpha_j}\rangle \overline{\sum_{j=1}^{n} \langle y, x_{\alpha_j}\rangle} \left\langle x_{\alpha_j}, x_{\alpha_j}\right\rangle$$

$$\|y\|^2 = \lim_{n\to\infty} \sum_{j=1}^{n} \left|\langle y, x_{\alpha_j}\rangle\right|^2 \left\langle x_{\alpha_j}, x_{\alpha_j}\right\rangle$$

As $\left\langle x_{\alpha_j}, x_{\alpha_j}\right\rangle = 1$,

$$\|y\|^2 = \lim_{n\to\infty} \sum_{j=1}^{n} \left|\langle y, x_{\alpha_j}\rangle\right|^2 \text{ as required.}$$

Lastly, the **converse statement**: If $\sum_{\alpha \in A} |c_\alpha|^2 < \infty, c_\alpha \in \mathbb{K}$, then $\sum_{\alpha \in A} c_\alpha x_\alpha$ converges to an element of $\mathcal{H}$.

Suppose for some $c_\alpha \in \mathbb{K}$,

$$\sum_{\alpha \in A} |c_\alpha|^2 < \infty$$

Consider the series $\sum_{\alpha\in A} c_\alpha x_\alpha$.

$$\sum_{\alpha\in A} |c_\alpha x_\alpha| \leq \sum_{\alpha\in A} |c_\alpha| \, \|x_\alpha\|$$
$$\sum_{\alpha\in A} |c_\alpha x_\alpha| \leq \sum_{\alpha\in A} |c_\alpha| \leq \sum_{\alpha\in A} |c_\alpha|^2 < \infty$$

Thus, $\sum_{\alpha\in A} c_\alpha x_\alpha$ is absolutely convergent and since $c_\alpha x_\alpha \in \mathcal{H}$, as $\mathcal{H}$ is complete then $\sum_{\alpha\in A} c_\alpha x_\alpha$ converges in $\mathcal{H}$. □

Remark 1.2. **Parseval's Relation**: (2) in the above theorem:

$$\|y\|^2 = \sum_{\alpha\in A} |\langle y, x_\alpha\rangle|^2$$

– Coefficients $\langle y, x_\alpha\rangle$ called *The Fourier Coefficients* of y with respect to the basis $\{x_\alpha\}_{\alpha\in A}$.

Recall: *Gram-Schmidt Orthogonalization*

– To construct an orthonormal set from an arbitrary sequence of independent vectors.

Suppose $\{u_1, u_2, \dots\}$ is an arbitrary set of independent vectors in an inner product space V. Construct an orthonormal set $\{v_1, v_2, \dots\}$ such that for each m, $\{u_j\}_{j=1}^m$ and $\{v_j\}_{j=1}^m$ span the same vector space.

– i.e., the set of all finite linear combinations of the v_i's is the same as the set of all finite linear combinations of the u_i's.

Recall the procedure:

$$
\begin{gathered}
w_1 = u_1 \longrightarrow v_1 = \frac{w_1}{\|w_1\|} \\
w_2 = u_2 - \langle v_1, u_1 \rangle v_1 \longrightarrow v_2 = \frac{w_2}{\|w_2\|} \\
\vdots \\
w_n = u_n - \sum_{k=1}^{n-1} \langle v_k, u_n \rangle v_k \longrightarrow v_n = \frac{w_n}{\|w_n\|} \\
\vdots
\end{gathered}
$$

and so on.

Recall:

Definition 1.3. A *separable Hilbert space* is a Hilbert space with a countable dense subset, i.e., there exists some countable $A \subset \mathcal{H}$ such that $\overline{A} = \mathcal{H}$.

– Most Hilbert spaces are separable

– Following theorem characterizes separable Hilbert spaces up to isomorphism.

Theorem II.7

A Hilbert space $\mathcal{H}$ is separable if and only if it has a countable orthonormal basis S.

- *If there are $N < \infty$ elements in S, then $\mathcal{H} \cong \mathbb{K}^N$.*
- *If there are countably many elements in S, then $\mathcal{H} \cong \ell^2$.*

Proof.

⌜ Before we start the proof, recall the following *Lemma* from Section I of the lectures (Normed Spaces):

Lemma: A normed space X is separable if and only if there exists a countable $A \subset X$ such that $X = \overline{\text{lin}}A$. ⌟

$\implies$ Suppose $\mathcal{H}$ is separable.

Let $A = \{a_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $\mathcal{H}$ ($\overline{A} = \mathcal{H}$) and hence $\overline{\text{lin}}A = \mathcal{H}$.

We can obtain a countable subcollection $V \subseteq A$ such that $V = \{v_i\}_{i\in\mathbb{N}}$ consists of only linearly independent vectors (i.e., discard the dependent vectors), such that the set of all finite linear combinations is the same as that of A.

$$\overline{\text{lin}}V = \overline{\text{lin}}A$$

Since $\overline{\text{lin}}A = \mathcal{H}$,

$$\Rightarrow \quad \overline{\text{lin}}V = \mathcal{H}$$

Hence, V is a subset of linearly independent vectors such that $\overline{\text{lin}}V = \overline{\text{lin}}A = \mathcal{H}$.

$V \subset \mathcal{H}$ is linearly independent, so by applying Gram-Schmidt, we obtain an orthonormal set S in $\mathcal{H}$ such that S is countable (as V is countable) and

$$\overline{\text{lin}}V = \overline{\text{lin}}S = \mathcal{H}$$

To show that S is an orthonormal basis, let $x \in \mathcal{H}$ be such that

$$\langle x, s_n \rangle = 0$$

for all $n \in \mathbb{N}$. (It must be shown that $x = 0$, because otherwise, $S \cup \{x\}$ would be an orthonormal set that properly contains S).

Let $S = \{s_n\}_{n\in\mathbb{N}}$.

As $\text{lin}S \subset \mathcal{H}$ is dense, let $\{w_k\}_{k\in\mathbb{N}} \subset \text{lin}S$ be such that

$$\lim_{k\to\infty} w_k = x$$

Since $\langle x, s_n \rangle = 0$ for all elements s_n of S, and since $w_k \in \text{lin}S$, then by the linearity and continuity of the inner product, the following holds.

$$\langle x, x \rangle = \lim_{k\to\infty} \langle x, w_k \rangle = 0$$

$$\langle x, x \rangle = 0 \quad \Rightarrow \quad x = 0 \text{ by property (1) of inner product}$$

Hence, if $x \in \mathcal{H}$ such that $\langle x, s_n \rangle = 0$ for all $n \in \mathbb{N}$, then $x = 0 \quad \Rightarrow \quad S$ is an orthonormal basis of $\mathcal{H}$.

So $\mathcal{H}$ has a countable orthonormal basis.

$\Leftarrow$ On the other hand, suppose $S = \{s_n\}_{n\in\mathbb{N}}$ is a countable orthonormal basis of $\mathcal{H}$.

By the previous theorem (II.6), for any $y \in \mathcal{H}$,

$$y = \sum_{i\in\mathbb{N}} \langle y, s_i \rangle s_i$$

$$\text{i.e., } y = \lim_{n\to\infty} \sum_{i=1}^{n} \langle y, s_i \rangle s_i$$

That is, $y \in \overline{\text{lin}}S$ (a limit of elements of $\text{lin}S$). As this holds for any $y \in \mathcal{H}$,

$$\overline{\text{lin}}S = \mathcal{H}$$

By the lemma: As S is countable and $\overline{\text{lin}}S = \mathcal{H}$, then $\mathcal{H}$ is separable.

• **If S has countably many elements**:

Suppose $\mathcal{H}$ is separable and $S = \{s_n\}_{n\in\mathbb{N}}$ is an orthonormal basis.

Define the map $\mathcal{U} : \mathcal{H} \to \ell^2$ by

$$\mathcal{U} : x \mapsto \{\langle x, s_n \rangle\}_{n\in\mathbb{N}}$$

– **$\mathcal{U}$ is well-defined and linear**:

By Theorem II.6 (previous theorem) for $x \in \mathcal{H}$,

$$x = \sum_{n=1}^{\infty} \langle x, s_n \rangle s_n \in \mathcal{H} \text{ such that}$$

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, s_n \rangle|^2$$

By the definition of norm, $\|x\|^2 < \infty$ and hence

$$\sum_{n=1}^{\infty} |\langle x, s_n \rangle|^2 < \infty$$

That is, for every $x \in \mathcal{H}$, $\mathcal{U}(x) \in \ell_2$. So $\mathcal{U}$ is well-defined.

Also, if $x, y \in \mathcal{H}, \alpha \in \mathbb{K}$,

$$\mathcal{U}(\alpha x + y) = \{\langle \alpha x + y, s_n \rangle\}_{n \in \mathbb{N}}$$
$$\mathcal{U}(\alpha x + y) = \{\langle \alpha x, s_n \rangle + \langle y, s_n \rangle\}_{n \in \mathbb{N}}$$
$$\mathcal{U}(\alpha x + y) = \{\alpha \langle x, s_n \rangle + \langle y, s_n \rangle\}_{n \in \mathbb{N}}$$
$$\Rightarrow \quad \mathcal{U}(\alpha x + y) = \alpha \mathcal{U}(x) + \mathcal{U}(y)$$

Thus, $\mathcal{U}$ is linear.

– $\mathcal{U}$ **is onto**:

Let $(c_j)_{j \in \mathbb{N}} \in \ell^2$ be arbitrary. Then,

$$\sum_{j=1}^{\infty} |c_j|^2 < \infty$$

By the second part of theorem II.6 (the converse statement), then $\sum_{n=1}^{\infty} c_j s_j$ converges to an element of $\mathcal{H}$. Thus, letting $x = \sum_{j=1}^{\infty} c_j s_j \in \mathcal{H}$,

$$\mathcal{U}(x) = \left\{ \left\langle \sum_{j=1}^{\infty} c_j s_j, s_n \right\rangle \right\}_{n \in \mathbb{N}}$$

By linearity (in 1st coordinate) of the inner product, then

$$\mathcal{U}(x) = \left\{ \sum_{j=1}^{\infty} c_j \langle s_j, s_n \rangle \right\}_{n \in \mathbb{N}}$$

Since $\langle s_j, s_n \rangle = 1$ iff $j = n$ (0 otherwise),

$$\mathcal{U}(x) = \{c_n\}_{n \in \mathbb{N}}$$
$$\Rightarrow \quad \mathcal{U}(x) = (c_j)_{j \in \mathbb{N}}$$

So $\mathcal{U}$ is onto.

– $\mathcal{U}$ **is isometric**:

Let $x \in \mathcal{H}$ be arbitrary.

It must be shown that $\|x\| = \|\mathcal{U}x\|$.

By definition,

$$\|\mathcal{U}x\|^2 = \langle \mathcal{U}x, \mathcal{U}x \rangle = \langle \langle x, s_n \rangle_{n \in \mathbb{N}}, \langle x, s_n \rangle_{n \in \mathbb{N}} \rangle$$

Using the inner product on ℓ^2,

$$\langle \mathcal{U}x, \mathcal{U}x \rangle = \sum_{n \in \mathbb{N}} \overline{\langle x, s_n \rangle} \langle x, s_n \rangle$$

$$\langle \mathcal{U}x, \mathcal{U}x \rangle = \sum_{n \in \mathbb{N}} |\langle x, s_n \rangle|^2$$

$$\Rightarrow \quad \langle \mathcal{U}x, \mathcal{U}x \rangle = \|x\|^2 \text{ by Theorem II.6}$$

$$\|\mathcal{U}x\|^2 = \|x\|^2$$

Therefore, as $x \in \mathcal{H}$ was arbitrary and $\|\mathcal{U}x\| = \|x\|$, $\mathcal{U}$ is isometric.

As $\mathcal{U}$ is well-defined, surjective, and isometric, then $\mathcal{U}$ is an isomorphism of Hilbert spaces.

$$\Rightarrow \quad \mathcal{H} \cong \ell^2$$

• **If S has N elements**: This is identical to the preceding argument for a countably infinite orthonormal basis S (replacing the inner product on ℓ^2 with the inner product on $\mathbb{K}^N$).

Suppose $\mathcal{H}$ is separable and $S = \{s_n\}_{n=1}^N$ is an orthonormal basis.

Define the map $\mathcal{U} : \mathcal{H} \to \mathbb{K}^N$ by

$$\mathcal{U} : x \mapsto \{\langle x, s_n \rangle\}_{n=1}^N$$

– $\mathcal{U}$ **is well-defined and linear**:

$\mathcal{U}$ is linear as in the previous case.

$\mathcal{U}$ is well-defined since for $x \in \mathcal{H}$:

$$\|x\|^2 = \sum_{n=1}^{N} |\langle x, s_n \rangle|^2 < \infty$$

Hence, $\mathcal{U}(x) \in \mathbb{K}^N$.

– $\mathcal{U}$ **is onto**:

Let $c = (c_1, c_2, \ldots, c_N) \in \mathbb{K}^N$ be arbitrary. Then,

$$\sum_{j=1}^{N} |c_j|^2 < \infty$$

By the second part of theorem II.6 (the converse statement), then $\sum_{j=1}^N c_j s_j$ converges to an element of $\mathcal{H}$. Thus, letting $x = \sum_{j=1}^N c_j s_j \in \mathcal{H}$,

$$\mathcal{U}(x) = \left\{ \left\langle \sum_{j=1}^{N} c_j s_j, s_n \right\rangle \right\}_{n=1}^{N}$$

By linearity (in 1st coordinate) of the inner product, then

$$\mathcal{U}(x) = \left\{ \sum_{j=1}^{N} c_j \langle s_j, s_n \rangle \right\}_{n=1}^{N}$$
$$\mathcal{U}(x) = \{c_n\}_{n=1}^N$$
$$\mathcal{U}(x) = (c_1, c_2, \ldots, c_N) = c$$

So $\mathcal{U}$ is onto.

– $\mathcal{U}$ **is isometric**:

Let $x \in \mathcal{H}$ be arbitrary.

It must be shown that $\|x\| = \|\mathcal{U}x\|$.

By definition,

$$\|\mathcal{U}x\|^2 = \langle \mathcal{U}x, \mathcal{U}x\rangle = \langle \langle x, s_n\rangle_{n=1}^N, \langle x, s_n\rangle_{n=1}^N\rangle$$

Using the standard inner product on $\mathbb{K}^N$,

$$\langle \mathcal{U}x, \mathcal{U}x\rangle = \sum_{n=1}^{N} \overline{\langle x, s_n\rangle}\langle x, s_n\rangle$$

$$\langle \mathcal{U}x, \mathcal{U}x\rangle = \sum_{n=1}^{N} |\langle x, s_n\rangle|^2$$

$$\Rightarrow \quad \langle \mathcal{U}x, \mathcal{U}x\rangle = \|x\|^2 \text{ by Theorem II.6}$$

$$\|\mathcal{U}x\|^2 = \|x\|^2$$

Therefore, as $x \in \mathcal{H}$ was arbitrary and $\|\mathcal{U}x\| = \|x\|$, $\mathcal{U}$ is isometric.

As $\mathcal{U}$ is well-defined, surjective, and isometric, then $\mathcal{U}$ is an isomorphism of Hilbert spaces.

$$\Rightarrow \quad \mathcal{H} \cong \mathbb{K}^N$$

□

Remark 1.4. Preceding theorem shows that in the separable case, the Gram-Schmdit process allows us to construct an orthonormal basis **without using Zorn's Lemma**.

How did Hilbert spaces naturally arise from problems in classical analysis?

If $f(x)$ is an integrable function on $[0, 2\pi]$, define c_n as

$$c_n = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^{2\pi} e^{-inx} f(x)dx$$

Definition 1.5. The formal series $\sum_{n=-\infty}^{\infty} c_n (2\pi)^{-\frac{1}{2}} e^{inx}$ is called the *Fourier series of f*.

Classical Problem: ***For which f and in what sense does the Fourier series of f converge to f?***

Theorem II.8

Suppose that $f(x)$ is periodic of period 2π and is continuously differentiable. Then the functions $\sum_{n=-M}^{M} c_n e^{inx}$ converge uniformly to $f(x)$ as $M \to \infty$.

– Gives sufficient conditions for Fourier series of a function to converge uniformly.

• But what about finding the exact class of functions whose Fourier series converge unfiromly or converge pointwise?

– Can get a nice answer to this question if we change notion of "convergence" and this is where we use Hilbert spaces

The collection of functions $\left\{(2\pi)^{-\frac{1}{2}}e^{inx}\right\}_{-\infty}^{\infty}$ is an *orthonormal set* in $L^2[0, 2\pi]$.

– If it was an *Orthonormal Basis*, then Theorem II.6 would imply that for all functions in $L^2[0, 2\pi]$,

$$f(x) = \lim_{M\to\infty} \sum_{n=-M}^{M} c_n(2\pi)^{-\frac{1}{2}}e^{inx}$$

i.e., convergence in L^2 norm.

Theorem II.9

If $f \in L^2[0, 2\pi]$, *then* $\sum_{n=-M}^{M} c_n(2\pi)^{-\frac{1}{2}}e^{inx}$ *converges to* f *in the* L^2 *norm as* $M \to \infty$.