CAYLEY GRAPHS AND EXPANDERS
PROJECT DOCUMENT

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1. Definitions and examples

Let Γ be a group and $\Sigma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \Gamma^k$ be a k-tuple of group elements.

Definition 1.1. The *directed Cayley graph* associated to (Γ, Σ) is given by

$$\mathrm{Cay}^+(\Gamma, \Sigma) = (\Gamma, \{e_{\gamma,i}\}_{\gamma,i}, e_{\gamma,i} \mapsto (\gamma, \gamma \cdot \gamma_i))$$

– $\mathrm{Cay}^-(\Gamma, \Sigma)$ is the *transpose* of Cay^+.

– $\mathrm{Cay}(\Gamma, \Sigma)$ is the *symmetrization* of Cay^+.

$\rightsquigarrow$ i.e., $\Sigma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \coprod \{\gamma_1^{-1}, \gamma_2^{-1}, \ldots, \gamma_k^{-1}\}$

Let $m \in \mathbb{Z}_{\geq 0}$.

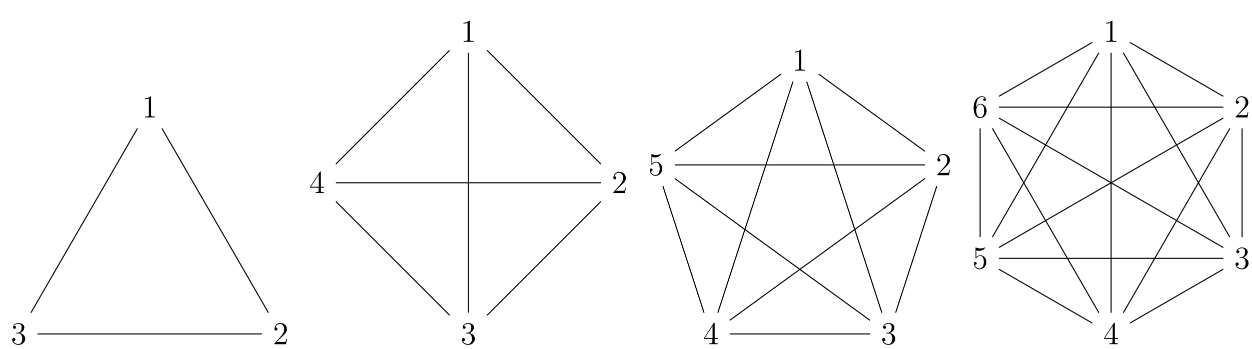
Definition 1.2. A *complete graph* K_m is a symmetric graph with $m \in \mathbb{Z}_{\geq 0}$ vertices such that there exists exactly one edge between every pair of distinct vertices, i.e.,

$$\mathrm{V}(K_m) = 1, 2, \ldots, m$$
$$\mathrm{E}(K_m) = \{(v, w) \mid \text{ where } v, w \in \mathrm{V}(K_m), v \neq w\}$$

The symmetrization of this graph (i.e., the "undirected" version is how this graph is usually visualized).

Remark 1.3. This is a simple graph because there is exactly one edge between any two *distinct* vertices.

Example 1.4. The symmetrizations of the complete graphs $K_3, K_4, K_5,$ and K_6.



Example 1.5.

$\text{Cycle}_m \cong \text{Cay}(\mathbb{Z}|m, \{\pm 1\})$ if and only if $m \geq 3$.

It is clear that $\text{Cycle}_1 \ncong \text{Cay}(\mathbb{Z}, \{\pm 1\}))$. Also, $\text{Cycle}_2 \ncong \text{Cay}(\mathbb{Z}|2, \{\pm 1\})$ since in $\mathbb{Z}|2, 1 = -1$ and $\text{Cay}(\mathbb{Z}|2, \{1\}) \cong \text{Path}_2$ but clearly $\text{Path}_2 \ncong \text{Cycle}_2$.

Hence, $\text{Cycle}_m \cong \text{Cay}(\mathbb{Z}|m, \{\pm 1\}) \Rightarrow m \neq 1, 2$. If $m \geq 3$, then $\text{Cay}(\mathbb{Z}|m, \{\pm 1\})$ is the graph with $\text{V(G)} = \mathbb{Z}|m$, and for every $v \in \text{V(G)}$, $\deg v = 2$, one edge $e \mapsto (v, v+1)$ and $e' \mapsto (v, v-1)$ (note: $\tau(e) \mapsto (v+1, v)$ and $\tau(e') \mapsto (v-1, v)$ by the symmetrization). Since this exactly describes a cycle graph on m vertices, then $\text{Cay}(\mathbb{Z}|m, \{\pm 1\}) \cong \text{Cycle}_m$.

Example 1.6. For all $m \geq 2$, $K_m \cong \text{Cay}(\mathbb{Z}|m, \mathbb{Z}|m \setminus \{0\})$.

This is clear by the definition of a complete graph K_m, as it has an edge between two vertices if and only if the vertices are distinct. Hence, construct a Cayley graph on the group $\mathbb{Z}|m$ by defining Σ as excluding $\{0\}$ from the vertex set (i.e., excluding self-loops). Then, this describes a complete graph on m vertices.

Remark 1.7. The above examples show that Cayley graphs can have a completely different structure even if the group Γ is the same.

Example 1.8. A combinatorial example.

Let $n \geq 3$. Define V_n as the finite set given by all possible arrangements of a deck of n cards, denote it D_n:

$$V_n = \{\text{all possible arrangements of a deck } D_n \text{ of } n \text{ cards}\}$$

By the multiplicative rule for counting, then $|V_n| = n!$. Suppose that the arrangement $a_1 a_2 \cdots a_n$ represents a_1 as the card on top and a_n as the card on the bottom. Define G_n as the symmetric simple graph on n vertices such that the symmetrized edges correspond to the following possibilities:

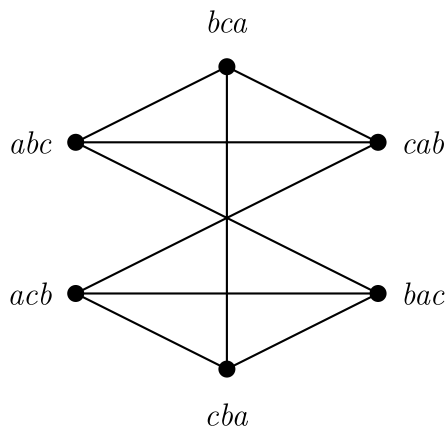
- Exchanging the top two cards;
- Bringing the bottom card to the top;
- Connecting the top to the bottom; or
- Connecting the bottom to the top.

To conceptualize this, consider a deck D_3 of 3 cards a, b, c. Then, $\text{V}_n = \{abc, acb, cba, bac, cab, bca\}$.

- Exchanging the top two cards identifies the following edges: $abc \leftrightarrow bac$, $acb \leftrightarrow cab$ and $cba \leftrightarrow bca$.
- Bringing the bottom card to the top identifies the following edges: $abc \leftrightarrow cab, acb \leftrightarrow bac, cba \leftrightarrow acb,$ and $bca \leftrightarrow abc$.
- Connecting the top to the bottom or the bottom to the top identifies the edges: $abc \leftrightarrow bca, acb \leftrightarrow cba, cba \leftrightarrow bac, bac \leftrightarrow acb, cab \leftrightarrow abc,$ and $bca \leftrightarrow cab$

Remark 1.9. In the above transformations of arrangements, if two edges are found between two arrangements, since G_n is a *simple* graph, we consider both transformations as representing a single edge (i.e., no multiple edges).

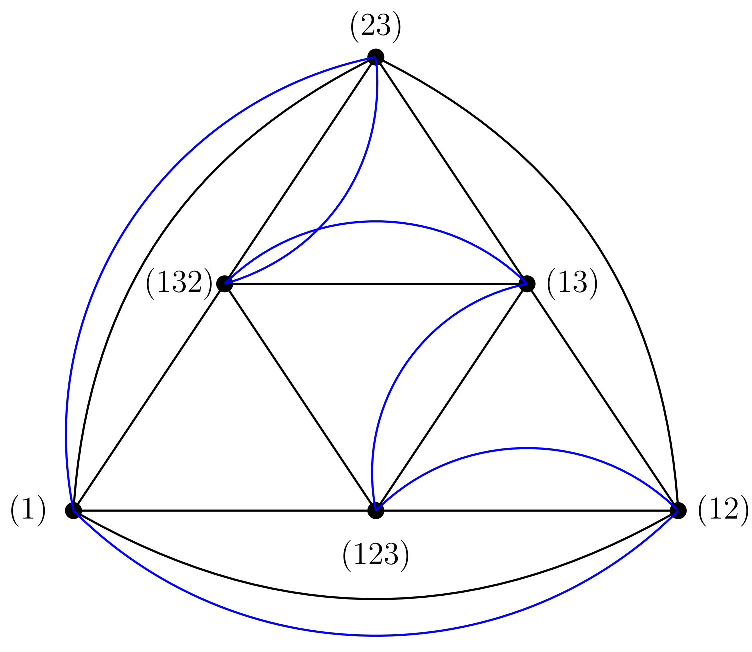
This graph G_3 would appear as follows (by drawing one single undirected edge to represent an edge e and its transpose $\tau(e)$).



Remark 1.10. This is isomorphic to the Cayley graph $\mathrm{Cay}(S_n, T_n)$ where S_n is the symmetric group and $T_n = \{\tau \mid \tau \in \mathrm{sym}([n]), |\tau| = 2\}$ (here, τ denotes any tranpsosition in the symmetric group S_n; i.e., exchanging any two elements).

Example 1.11. A symmetric group example of a Cayley graph.

$\mathrm{Cay}(S_3, \Sigma)$ where $\Sigma = \{(12), (23), (123)\} \cup \{(12), (23), (132)\}$ (note: contains multiple edges).



Proposition 1.12. $\mathrm{Cay}(\Gamma, \Sigma)$ *is* $2k$*-regular.*

Proof. This follows by definition of a Cayley graph and the fact that

$$\Sigma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \coprod \{\gamma_1^{-1}, \gamma_2^{-1}, \ldots, \gamma_k^{-1}\}.$$

That is, for any vertex $v \in \Gamma$, there are $2k$ choices for an edge with their tail at v since $(t, h)(e) = (v, w)$ if and only if $w = v \cdot \gamma_i$ for some $\gamma_i \in \Sigma$. There are $2k$ choices for γ_i. As there exist exactly $|\Sigma| = 2k$ edges with their tail at v then by symmetry, $\deg v = 2k \Rightarrow \mathrm{Cay}(\Gamma, \Sigma)$ is $2k$-regular. □

Proposition 1.13. $\mathrm{G} = \mathrm{Cay}(\Gamma, \Sigma)$ *is connected if and only if* Σ *is a generating set of* Γ.

Proof.

$\Rightarrow$ Suppose G is connected. Let $x \in \Gamma = \mathrm{V(G)}$ be arbitrary.

Since G is connected, there exists some path of length $m \in \mathbb{Z}_{\geq 0}$ from $1 \in \Gamma$ to $x \in \Gamma$. For this m, let $f : \mathrm{Path}_m \to \mathrm{G}$ be a graph morphism such that $\mathrm{V}(f)(0) = 1$ and $\mathrm{V}(f)(m-1) = x$. For $0 \leq i \leq m-1$, denote $x_i = \mathrm{V}(f)(i) \in \mathrm{V(G)}$. Then, by definition of f and x_i, we have that

$$x_0 = 1 \text{ and } x_m = x$$

By definition of edges in $\mathrm{G} = \mathrm{Cay}(\Gamma, \Sigma)$, for each i, there exists some $\gamma_i \in \Sigma$ such that

$$\mathrm{E}(f)(i) = (x_{i-1}, x_{i-1} \cdot \gamma_i) = (x_{i-1}, x_i)$$

$$\Rightarrow \quad x_i = x_{i-1} \cdot \gamma_i$$

By induction (and the definition of f and Path_m), then the following holds.

$$x = x_m = x_{m-1} \cdot \gamma_m = (x_{m-2} \cdot \gamma_{m-1}) \cdot \gamma_m = \cdots = \gamma_1 \cdot \gamma_2 \cdot \gamma_3 \cdots \gamma_m$$

Hence, $x \in \langle \Sigma \rangle$. As $x \in \Gamma$ was arbitrary, then $\Gamma \subseteq \langle \Sigma \rangle$ and hence $\langle \Sigma \rangle = \Gamma$. Σ is a generating set of Γ.

$\Leftarrow$ On the other hand, suppose Σ is a generating set of the group Γ.

Let $x, y \in \Gamma = \mathrm{V(G)}$ be arbitrary vertices of the cayley graph $\mathrm{Cay}(\Gamma, \Sigma)$. By definition of groups, $x^{-1}y \in \Gamma$. Since $\Gamma = \langle \Sigma \rangle$, then there exists some $d \in \mathbb{Z}_{>0}$ such that

$$x^{-1}y = \gamma_1 \cdot \gamma_2 \cdots \gamma_d$$

Define $g : \mathrm{Path}_d \to \mathrm{G}$ as the graph morphism given by the following.

$$\mathrm{V}(g)(0) = x;$$

$$\mathrm{V}(g)(i) = x \cdot \gamma_1 \gamma_2 \cdots \gamma_i \text{ for all } 1 \leq i \leq d; \text{ and}$$

$$\mathrm{E}(g)(i-1, i) = (x \cdot \gamma_1 \gamma_2 \cdots \gamma_{i-1}, x \cdot \gamma_1 \gamma_2 \cdots \gamma_i) \text{ for all } 1 \leq i \leq d$$

This is a well-defined graph morphism since there exists an edge from the vertex $x \cdot \gamma_1 \gamma_2 \cdots \gamma_{i-1}$ to $x \cdot \gamma_1 \gamma_2 \cdots \gamma_{i-1} \gamma_i$ for all i (simply by the Cayley structure of the graph and multiplying by $\gamma_i \in \Sigma$).

Note that this morphism shows that $\mathrm{V}(g)(0) = x$ and $\mathrm{V}(g)(d) = x \cdot \gamma_1 \cdot \gamma_2 \cdots \gamma_d = x \cdot (x^{-1}y) = y$. Hence, there exists a path of length $d > 0$ from x to y. As $x, y \in \mathrm{V(G)}$ were arbitrary, then G is connected.

□

Proposition 1.14. *Suppose Σ is a generating set of the group Γ.*
$\mathrm{Cay}(\Gamma, \Sigma)$ *is bipartite $\Leftrightarrow$ there exists a surjective group homomorphism $\pi : \Gamma \to \{\pm 1\}$ such that $\pi(\gamma_i) = -1$ for all $\gamma_i \in \Sigma$.*

Proof.

$\Rightarrow$ Suppose that $\mathrm{Cay}(\Gamma, \Sigma)$ is bipartite. By definition, there exists a map (a colouring) $c : \Gamma \to \{\pm 1\}$ such that for all $v \sim w \in \Gamma$, $c(v) \neq c(w)$. Considering $c(1)$, if $c(1) \neq 1$, then $-c$ is also a colouring of the vertices Γ. Without loss of generality, assume that $c(1) = 1$.

Claim 1.15. *c is a group homomorphism.*

This can be shown by inductively taking steps away from $1 \in \Gamma$. If $1 \sim \gamma$, then by the colouring c, $c(\gamma) = -1$ and hence $c(\gamma) = c(\gamma \cdot 1) = -1 = c(\gamma) \cdot c(1)$. Continuing in this way will show that c is a group homomorphism such that $c(\gamma) = (-1)^k$ where k is the minimal path length from γ to 1 (note: $\mathrm{Cay}(\Gamma, \Sigma)$ is connected so this is well-defined). Then, for any $\gamma, \gamma' \in \Gamma$, $c(\gamma \cdot \gamma') = c(\gamma) \cdot c(\gamma')$. Also, since $1 \sim \gamma_i$ for all i (by the tail-head map of the Cayley graph), then by this colouring/homomorphism: $c(\gamma_i \cdot 1) \neq c(1) = 1$, hence $c(\gamma_i) = -1$ for all i.

$\Leftarrow$ Conversely, suppose there exists a homomorphism $\pi : \Gamma \to \{\pm 1\}$ such that $\pi(\gamma_i) = -1$ for all $\gamma_i \in \Sigma$. Then, for any $\gamma \in \Gamma$, $\pi(\gamma \cdot \gamma_i) = \pi(\gamma) \cdot \pi(\gamma_i)$. This implies that $\pi(\gamma \cdot \gamma_i) = -\pi(\gamma) \neq \pi(\gamma)$. That is, for any $\gamma \in \Gamma$, $\pi(\gamma) \neq \pi(\gamma \cdot \gamma_i)$. In terms of vertices, then this is equivalent to stating that for any vertices of the Cayley graph $v, w \in \mathrm{V}(\mathrm{Cay}(\Gamma, \Sigma))$, if $v \sim w$, then $\pi(v) \neq \pi(w)$. By definition, then π is a colouring of the vertices of $\mathrm{Cay}(\Gamma, \Sigma)$ and thus $\mathrm{Cay}(\Gamma, \Sigma)$ is bipartite. □

Remark 1.16. The preceding proposition implies the following result: If $1 \in \Sigma$ then $\mathrm{Cay}(\Gamma, \Sigma)$ is not bipartite. i.e., $\pi(1 \cdot 1) = \pi(1) = \pi(1)^2$ and since $(-1) \neq (-1)^2$, then $\pi(1) \neq -1$.

2. Automorphisms of Cayley graphs and Cayley action graphs

Let Γ be a group, $\Sigma = (\gamma_1, \gamma_2, \ldots, \gamma_k) \in \Gamma^k$ be an arbitrary k-tuple of group elements as before, and let $\mathrm{G} = \mathrm{Cay}(\Gamma, \Sigma)$ be the Cayley graph on (Γ, Σ).

Consider an arbitrary group action of a group M on a set X, $M \times X \to X$.

Definition 2.1.

(1) The group action is *faithful* if for every non-identity $m \in M$, there exists some $x \in X$ such that $m \cdot x \neq x$.

(2) The group action is *transitive* if for every $x, y \in X$, there exists some $m \in M$ such that $m \cdot x = y$.
   - i.e., the group action has a single orbit.

Definition 2.2. An *automorphism* of $\mathrm{Cay}(\Gamma, \Sigma)$ is a bijective graph morphism $f : \mathrm{Cay}(\Gamma, \Sigma) \to \mathrm{Cay}(\Gamma, \Sigma)$. The set of graph automorphisms of G forms a group, called the *Cayley automorphism group* of G, denote this by Γ_{aut}.

Definition 2.3. A *group action of* Γ *by graph automorphisms* on $\mathrm{Cay}(\Gamma, \Sigma)$ is a group action $\Gamma_{aut} \times \mathrm{Cay}(\Gamma, \Sigma) \to \mathrm{Cay}(\Gamma, \Sigma)$.

– i.e., $(f, v) \mapsto \mathrm{V}(f)(v)$ and $(f, e) \mapsto \mathrm{E}(f)(e)$ for all $v \in \mathrm{V}(\mathrm{G}), e \in \mathrm{E}(\mathrm{G})$ and where $f \in \mathrm{Hom}(\mathrm{Cay}(\Gamma, \Sigma))$ is bijective.

Proposition 2.4. *The automorphism group* Γ_{aut} *acts faithfully and transitively on* $\mathrm{Cay}(\Gamma, \Sigma)$*; the automorphism* f_g *associated to* $g \in \Gamma$ *is given by*

$$V(f_g)(x) = gx$$
$$E(f_g)\,(x, x\gamma_i) = (gx, gx\gamma_i)$$

for all vertices $x \in \Gamma$ *and edges* $(x, x\gamma_i) \in E(\mathrm{Cay}(\Gamma, \Sigma))$.

Proof. First note that this group action of Γ by graph automorphisms on $\mathrm{G} = \mathrm{Cay}(\Gamma, \Sigma)$ acts by sending $f_g \mapsto \{V(f_g)(\mathrm{G}), E(f_g)(\mathrm{G})\} \subset \mathrm{G}$.

It is faithful since for any $f_g \neq \mathrm{id}_{\Gamma_{aut}}$, then $V(f_g)(x) \neq x$ for $x \neq 1$ by definition of f_g given above.

Also, it is transitive since for each $x, y \in \mathrm{V}(\mathrm{G})$, define $g = yx^{-1}$, then $\mathrm{V}(f_{yx^{-1}})(x) = yx^{-1}x = y$ by definition. □

Remark 2.5. The above proposition is equivalent to saying that Γ acts by graph automorphisms on $\mathrm{Cay}(\Gamma, \Sigma)$ and in particular, every vertex "looks the same."

Let $\Gamma \times \Omega \to \Omega$ be a left action of Γ on an arbitrary set Ω.

Definition 2.6. An *directed action graph* of Ω with respect to Σ, denoted $\mathcal{A}^+(\Omega, \Sigma)$ is defined by the following.

$$\mathrm{V}(\mathcal{A}^+(\Omega,\Sigma)) = \Omega$$
$$\mathrm{E}(\mathcal{A}^+(\Omega,\Sigma)) = (\Sigma \times \Omega)/\sim$$
$$(t,h)(e) = (\gamma_i \cdot \omega, \omega) \text{ for } e \in \mathrm{E}(\mathcal{A}^+(\Omega,\Sigma))$$

where $\sim$ is an equivalence relation on $\Sigma \times \Omega$ given by the following for all $\gamma_i \in \Sigma$ and $\omega \in \Omega$:

$$(\gamma_i, \omega) \sim (\gamma_i, \omega) \text{ and}$$
$$\gamma_i \cdot \omega \neq \omega \Rightarrow (\gamma_i, \omega) \sim (\gamma_i^{-1}, \gamma_i \cdot \omega)$$

Let $\Theta \leq \Gamma$ be a subgroup of Γ.

Definition 2.7. The *Schreier graph* of Γ/Θ with respect to Σ is the directed action graph $\mathcal{A}^+(\Gamma/\Theta, \Sigma)$ where Γ acts on Γ/Θ by $g \cdot (x\Theta) = (gx)\Theta$ for all $g \in \Gamma, x\Theta \in \Gamma/\Theta$.

Definition 2.8. The *relative Cayley graph* of Γ of Γ/Θ with respect to Θ is the Schreier graph of Γ/Θ with respect to Σ when $\Theta \trianglelefteq \Gamma$ is normal. This is denoted by $\mathcal{C}^+(\Gamma/\Theta, \Sigma)$.

Lemma 2.9. *For any $\omega \in \Omega$, there is a natural identification between Σ and the set of edges $\alpha \in \mathcal{A}^+(\Omega, \Sigma)$ with either $t(\alpha)$ or $h(\alpha)$ equal to ω. Specifically, $\mathcal{A}^+(\Omega, \Sigma)$ is k-regular.*

Proof.

Fix $\omega \in \Omega$. Denote $\mathrm{G} = \mathcal{A}^+(\Omega, \Sigma)$. Define $E_\omega \subset \mathrm{E}(\mathrm{G})$ by the following.

$$E_\omega = \mathrm{E}(\mathrm{G}; {}^{\to}\omega) \cup \mathrm{E}(\mathrm{G}; \omega^{\to})$$

Define a map $\varphi : \Sigma \to E_\omega$ by $\varphi : \gamma_i \mapsto [(\gamma_i, \omega)]$ for all $\gamma_i \in \Sigma$ where $[(\gamma_i, \omega)]$ denotes the equivalence class of (γ_i, ω) with respect to $\sim$ as in the definition of the action graph.

Claim: φ is a bijection.

First, φ is injective since if $\varphi(\gamma_i) = \varphi(\gamma_j)$, then $(\gamma_i, \omega) \sim (\gamma_j, \omega)$. By definition of this equivalence relation, either $\gamma_i = \gamma_j$ or $\gamma_i\omega \neq \omega$ implies that $(\gamma_j, \omega) = (\gamma_i^{-1}, \gamma_i\omega)$ but this is a contradiction since it implies that $\omega = \gamma_i\omega$. Hence, it follows that $\gamma_i = \gamma_j$. i.e., $\varphi(\gamma_i) = \varphi(\gamma_j) \Rightarrow \gamma_i = \gamma_j$ for all $\gamma_i, \gamma_j \in \Sigma$, so φ is injective.

Also, φ is surjective. Let $\alpha \in E_\omega$ be given by $\alpha = [(\gamma_j, y)]$ for some $\gamma_j \in \Sigma, y \in \Omega$. Since $\alpha \in E_\omega$, by definition, then either $t(\alpha) = \omega$ or $h(\alpha) = \omega$.

If $h(\alpha) = \omega$, then by $\alpha = [(\gamma_j, y)]$, this implies that $\omega = y$, thus $\alpha = \varphi(\gamma_j)$.

If $t(\alpha) = \omega$, then by the definition of the tail-head map of the action graph, $\omega = \gamma_j y \neq y$ and the equivalence relation $\sim$ implies the following.

$$(\gamma_j, y) \sim (\gamma_j^{-1}, \gamma_j \cdot y)$$

$$\Rightarrow (\gamma_j, y) \sim (\gamma_j^{-1}, \gamma_j \cdot y) = (\gamma_j^{-1}, \omega)$$
$$\Rightarrow \quad \alpha = \varphi(\gamma_j^{-1})$$

In both cases, there exists some $\gamma \in \Sigma$ such that $\varphi(\gamma) = \alpha$. Thus, φ is surjective. Hence, for each $\omega \in \Omega$, there exists a bijection between Σ and E_ω, and thus $\mathrm{G} = \mathcal{A}^+(\Omega, \Sigma)$ is $|\Sigma| = k$-regular. $\square$

3. Expanders: General

Let $\mathrm{G} = (\mathrm{V}, \mathrm{E}, \mathrm{E} \xrightarrow{(t,h)} \mathrm{V}^2)$ be a finite symmetric k-regular graph $\quad \rightsquigarrow \quad |\,\mathrm{V}\,| = n$

Let $X \subset \mathrm{V}$ and $\overline{X} = \mathrm{V} \setminus X$.

Definition 3.1. Recall the following definition.

$\mathrm{E}(\mathrm{G}; X, \overline{X}) = \coprod_{x \in X} \coprod_{y \in \overline{X}} \mathrm{E}(\mathrm{G}; x, y)$ where $\mathrm{E}(\mathrm{G}; x, y)$ is the set of edges from vertex x to vertex y.

- i.e., $\mathrm{E}(\mathrm{G}; x, y) = \{e \in \mathrm{E} \mid t(e) = x, h(e) = y\}$.

Definition 3.2. The *expansion constant* $h(\mathrm{G})$ of a finite graph is

$$h(\mathrm{G}) = \min_{X \subsetneq V, X \neq \emptyset} \left\{ \frac{|\,\mathrm{E}(\mathrm{G}; X, \overline{X})|}{\min\{|X|, |\overline{X}|\}} \right\}$$

$$\Rightarrow \quad h(\mathrm{G}) = \min_{X \subsetneq V, X \neq \emptyset} \left\{ \frac{|\,\mathrm{E}(\mathrm{G}; X, \overline{X})|}{|X|} \middle| \, |X| \leq \frac{n}{2} \right\}$$

One could ask why the former implies the latter and why we only look at subsets of size less than or equal to $\frac{n}{2}$. Suppose that $X \subset \mathrm{V}$ and $|X| > \frac{n}{2}$. Then, clearly $|\overline{X}| \leq \frac{n}{2}$ and hence $\min\{|X|, |\overline{X}|\} = |\overline{X}|$. i.e., "$Y$" $= \overline{X} \subset \mathrm{V}$ is already considered in the above definition of the expansion constant $h(\mathrm{G})$ since $|Y| \leq \frac{n}{2}$. So it makes sense to only consider the subsets of V that have size less than $\frac{n}{2}$, since all others are accounted for by taking the minimum of the sizes of a subset and its respective complement.

Corollary 3.3. $h(\mathrm{G}) > 0 \Leftrightarrow \mathrm{G}$ *is connected.*

Let $\{\mathrm{G}\}_{i \in I}$ be a family of finite symmetric graphs.

Definition 3.4. $\{\mathrm{G}\}_{i \in I}$ is a *weak (one-sided) expander family* if the following conditions hold.

1. $|\,\mathrm{V}_i\,| \to \infty$
   - i.e., for all $n \geq 1$, there exist only finitely many $i \in I$ such that $|\,\mathrm{V}_i\,| \leq n$.
2. $\liminf_{i \in I} h(\mathrm{G}_i) \geq h$ for some $h > 0$.
3. $\max\{\deg(v) \mid i \in I \text{ and } v \in \mathrm{V}(\mathrm{G}_i)\} \leq k$ for some $k > 0$.

Let $\{\mathrm{G}\}_{i \in I}$ be a family of finite, symmetric, *and* k-regular graphs where $|\,\mathrm{V}(\mathrm{G}_i)| = n_i$ for each $i \in I$.

Definition 3.5. $\{\mathrm{G}\}_{i \in I}$ an *absolute expander family* (or *strong* expander) if the following conditions hold.

(1) $|\,\mathrm{V}_i\,| \to \infty$

(2) $\limsup_{i \in I} \max\{|\alpha_2(\mathrm{G}_i)|, |\alpha_{n_i}(\mathrm{G}_i)\} < k$
where $\mathrm{spec}(A(\mathrm{G}_i)) = \{\alpha_1(\mathrm{G}_i) \geq \alpha_2(\mathrm{G}_i) \geq \cdots \geq \alpha_{n_i}(\mathrm{G}_i)\}$ is the set of eigenvalues of the adjacency matrix of the graph G_i in decreasing order.

4. Cayley graphs as expanders

Let Γ be a group, $\Sigma = (\gamma_1, \gamma_2, \dots, \gamma_k) \in \Gamma^k$ be an arbitrary k-tuple of group elements as before, and let $\mathrm{G} = \mathrm{Cay}(\Gamma, \Sigma)$ be the Cayley graph on (Γ, Σ).

Definition 4.1. The *distance* between any two vertices $x, y \in \mathrm{V}$ is the minimum length of a path between x and y, denoted $\mathrm{dist}_{\mathrm{G}}(x, y)$; if no path exists between x and y, define $\mathrm{dist}_{\mathrm{G}}(x, y) = \infty$. If $\mathrm{dist}_{\mathrm{G}}(x, y) \neq \infty$, then

$$\mathrm{dist}_{\mathrm{G}}(x, y) = \min_{\varphi \in \mathrm{Hom}_\tau(\mathrm{Path}_d^+, \mathrm{G})} \{d \mid \varphi(0) = x, \varphi(d) = y\}$$

– For any $x, y \in \mathrm{V}$, $\mathrm{dist}_{\mathrm{G}}(x, y) \in \{0, 1, \dots\} \cup \infty$.

Remark 4.2. dist is a metric on V and thus $(\mathrm{V}(\mathrm{G}), \mathrm{dist}_{\mathrm{G}})$ is a metric space.

Remark 4.3. The definitions of distance and diameter of graphs holds for *any* finite graph, not just Cayley graphs.

Definition 4.4. The *diameter* of G is the largest distance between any two vertices.

$$\mathrm{diam}(\mathrm{G}) = \begin{cases} \sup_{x,y \in \mathrm{V}} \mathrm{dist}_{\mathrm{G}}(x, y) < \infty & \text{if G is connected} \\ \infty & \text{if G is disconnected} \end{cases}$$

Remark 4.5. Recall the following definition of *"big-oh notation"*: Let $f, g : \mathbb{N} \to \mathbb{R}^+$. $f(n) = O(g(n))$ iff there exist constants $N, c > 0$ such that $f(n) \leq cg(n)$ for all $n > N$. i.e., at some point, f is bounded above by a constant times g.

Let $\{\mathrm{G}_n\}_{n \in N}$ be a sequence of finite graphs.

Definition 4.6. The sequence of finite graphs $\{\mathrm{G}_n\}_{n \in N}$ has *logarithmic diameter* iff

$$\mathrm{diam}(\mathrm{G}_n) = O(\log |\, \mathrm{G}_n \,|)$$

Let $\{\Gamma_n\}_{n \in \mathbb{N}}$ be a sequence of finite groups.

Definition 4.7. $\{\Gamma_n\}_{n \in \mathbb{N}}$ has *logarithmic diameter* if there exists some $d \in \mathbb{Z}_{>0}$ such that there exists a sequence $\{\Phi_n\}_{n \in \mathbb{N}}$ where for each n, $\Phi_n \subset \Gamma_n$ is a symmetric subset with $|\Phi_n| = d$, so that the sequence of Cayley graphs $(\mathrm{Cay}(\Gamma_n, \Phi_n))_{n \in \mathbb{N}}$ has logarithmic diameter.

Let Γ be a group, $\Sigma = (\gamma_1, \gamma_2, \dots, \gamma_k) \in \Gamma^k$ be an arbitrary k-tuple of group elements as before.

Definition 4.8. A *word of length n* in Σ is an element of the direct product Σ^n. If $w = (w_1, \dots, w_n) \in \Sigma^n$, then w *evaluates to* g (or g *can be expressed as* w) iff $g = w_1 \cdot w_2 \cdots w_n$.

Example 4.9. Let $\Gamma = \mathbb{Z}$ and $\Sigma = \{-1, 1, 2, 5\}$.

Then, $w = (1, 2, 1)$ is a word of length 3 in Σ and $w' = (2, 2)$ is a word of length 2. Since $4 = 1 + 2 + 1 = 2 + 2$, both w and w' evaluate to 4. That is, an element of the group can be expressed non-uniquely as a word in Σ.

Let $g \in \Gamma$ be such that it can be expressed as a word in Σ.

Definition 4.10. The *word norm of g in Σ* is the minimal length of any word in Σ which evaluates to g.

- Convention $\rightsquigarrow$ Word of length 0 evaluates to the identity.

Using the assumptions at the beginning of this section: Γ is a group, $\Sigma \subset \Gamma$ be an arbitrary k-tuple of group elements, and $\mathrm{G} = \mathrm{Cay}(\Gamma, \Sigma)$.

Proposition 4.11.

(1) G *is connected if and only if every element of* Γ *can be expressed as a word in* Σ.

(2) *If* $a, b \in \Gamma$ *and there exists a path from* a *to* b*, then* $\mathrm{dist}_{\mathrm{G}}(a, b) =$ *word norm of* $a^{-1}b \in \Sigma$.

(3) $\mathrm{diam}(\mathrm{G})$ *equals the maximum of the word norms in* Σ *of elements of* Γ.

Proof.

(1) This is exactly equivalent to saying G is connected $\Leftrightarrow$ Σ generates Γ, as shown in Proposition 1.13

(2) Suppose there exists a path of length d from a to b, i.e., $\mathrm{Hom}_\tau(\mathrm{Path}_d^+, \mathrm{G}) \neq \varnothing$. Let $(g_0, g_1, \ldots, g_d)$ denote the vertices of this path in G. That is, $a = g_0, b = g_d$.
Define
$$\gamma_j = g_{j-1}^{-1} g_j \text{ for all } j = 1, 2, \ldots, d$$
By definition of of the $(g_0, \ldots, g_d)$ path and the Cayley graph G on (Γ, Σ), then $\gamma_j \in \Sigma$ for all j and there exists an edge in the Cayley graph from g_{j-1} to g_j by definition. Then, the following holds by construction.
$$\gamma_1 \gamma_2 \cdots \gamma_d = g_0^{-1} g_1 g_1^{-1} g_2 \cdots g_{d-1}^{-1} g_d$$
$$\gamma_1 \gamma_2 \cdots \gamma_d = g_0^{-1} g_d = a^{-1} b$$
Thus, $(\gamma_1, \gamma_2, \ldots, \gamma_d)$ is a word of length d in Σ that evaluates to $a^{-1}b$.

On the other hand, every word of length d in Σ that evaluates to $a^{-1}b$ can be associated with a path of length d in G from a to b. This is done by taking each element of the word as a vertex in the path. Now, $\mathrm{dist}_{\mathrm{G}}(a, b)$ is the minimal length of all possible paths from a to b, which by the given correspondence, equals the minimal length of all words in Σ that evaluate to $a^{-1}b$. By definition, this is equivalent to the word norm of $a^{-1}b$ in Σ.

(3) If $g \in \Gamma$, then by part (2) of this proposition, $\mathrm{dist}_{\mathrm{G}}(e, g)$ is the word norm of g. By definition of diam(G) as a supremum, then

$$\mathrm{diam(G)} \geq \max_{g \in \Gamma} \mathrm{dist}_{\mathrm{G}}(e, g)$$

So diam(G) is greater than the maximum of the word norms of all $g \in \Gamma$. By (2), *every* distance is a word norm, and hence diam(G) attains the maximum and is equal to the maximum of the word norms in Σ of elements of Γ.

□

Remark 4.12. Part (3) of the above proposition asserts that the *word norm* of a group element $g \in \Gamma$ equals the distance from the identity element e to g in $\mathrm{G} = \mathrm{Cay}(\Gamma, \Sigma)$.

Motivation for constructing expanders: What assumptions are needed to conclude that a family of Cayley graphs is an expander?

Theorem 1. *Expansion in Subgroups of $SL_2(\mathbb{Z})$*

Let $\Sigma \subset SL_2(\mathbb{Z})$ be any finite subset, $\Theta = \langle \Sigma \rangle$ be the subgroup generated by Σ, and let p be a prime. Define

$$\mathrm{G}_p = \mathrm{Cay}(SL_2(\mathbb{F}_p), \Sigma)$$

as the Cayley action graph of the finite quotient group $SL_2(\mathbb{F}_p)$ with respect to the reduction modulo p of the set Σ.

Then

$$\{\mathrm{G}_p\}_{p \geq p_0} \text{ is expander } \Leftrightarrow \mathrm{G} \text{ is connected for all } p \geq p_0 \text{ and some } p_0$$

Corollary 4.13. *Let $k \geq 1, k \in \mathbb{Z}$. Define the subset $\Sigma \subset SL_2(\mathbb{Z})$ as follows.*

$$\Sigma = \left\{ \begin{pmatrix} 1 & \pm k \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm k & 1 \end{pmatrix} \right\} \subset SL_2(\mathbb{Z})$$

For a prime p, let $\Sigma_p :=$ the image of Σ modulo p.

If $p \nmid k$, then $\mathrm{Cay}(SL_2(\mathbb{F}_p), \Sigma_p)$ is connected, so $(\mathrm{Cay}(SL_2(\mathbb{F}_p), \Sigma_p))_{p \nmid k}$ is expander.

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