II.11.2021

Higher Discrete Homotopy Groups of Graphs

Sources:
- "Higher discrete homotopy groups of graphs" [LUTZ]
- "Foundations of a connectivity theory for simplicial complexes" [BARCELO, KRAMER, LAUBENBACHER, WEAVER]

1. Notation, definitions
2. Contractible graphs have trivial homotopy groups
3. If G has no 3- or 4-cycles, then $\pi_n(G)$ is trivial $\forall n \geq 2$
4. Broad overview of a discrete Hurewicz theorem.

§ Notation, definitions

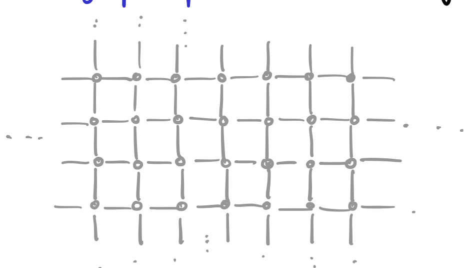
Recall the following definitions: $G=(V,E)$, $G_1=(V_1,E_1)$, and $G_2=(V_2,E_2)$ simple graphs

- A graph map $f: G_1 \to G_2$ is a map on vertices $V_1 \to V_2$ such that if $v \sim w$ in E_1, then

$$f(v)=f(w) \quad \text{or} \quad f(v) \sim f(w)$$

- I_∞ : graph with vertex set $V(I_\infty) := \mathbb{Z}$ and an edge $i \sim j \Leftrightarrow |i-j|=1$
- I_∞^n : the n-fold graph product of the graph I_∞ ; $I_\infty^n := \underbrace{I_\infty \otimes I_\infty \otimes \cdots \otimes I_\infty}_{n \text{ times}}$

(a portion of) I_∞^2



- $I_{\geq r}^n$: subgraph of I_∞^n induced by vertices $x \in I_\infty^n$ such that $|x_i| \geq r$ for some $1 \leq i \leq n$.
  - usually will be used to refer to the "dead area" of a certain loop $f: I_\infty^n \to G$.
- Two graph maps $f,g: G_1 \to G_2$ are homotopic if for some $m \geq 0$, there is a graph map (a homotopy) $h: G_1 \otimes I_m \to G_2$ such that

$$h(-,0)=f \quad \text{and} \quad h(-,m)=g$$

- Two graph maps $f,g: (I_\infty^n, I_{\geq r}^n) \to (G, v_0)$ are based homotopic if there is a homotopy $h: I_\infty^n \otimes I_m \longrightarrow G$ from f to g such that $h_i := h(-,i)$ is a graph map $h_i : (I_\infty^n, I_{\geq r_i}^n) \longrightarrow (G, v_0)$ for all i.

  $\rightsquigarrow$ Denote the homotopy class by $[f]$

- The radius of a "loop" $f:(I^n_\infty, I^n_{\geq r}) \to (G, v_0)$ is the minimum such $r \geq 0$, sometimes denoted r_f.

  i.e. For all $x \in \mathbb{Z}^n$ where $|x_i| \geq r$ for some i, $f(x) = v_0$.

Def The n^{th} discrete homotopy group of G, $\pi_n(G)$: Fix $v_0 \in G$.

$$\pi_n(G) := \left\{ [f] \;\middle|\; f:(I^n_\infty, I^n_{\geq r}) \to (G, v_0) \text{ a graph map} \right\}$$

- Group structure: multiplication = concatenation (in the first variable)
  - $f:(I^n_\infty, I^n_{\geq r_f}) \to (G, v_0)$
  - $g: (I^n_\infty, I^n_{\geq r_g}) \to (G, v_0)$

  $\rightsquigarrow$ $[f]\cdot[g] = [p]$ where $p:(I^n_\infty, I^n_{\geq r}) \longrightarrow (G, v_0)$ is defined by

$$p(x_1, x_2, \dots, x_n) := \begin{cases} f(x_1, \dots, x_n) & \text{if } x_1 \leq r_f \\ g(x_1 - (r_f + r_g), x_2, \dots, x_n) & \text{if } x_1 > r_f \end{cases}$$

eg In $\pi_2(G)$

  - identity: the class of the constant v_0 map.

Some facts: [BARCELO ET AL.]

1. The operation in $\pi_n(G)$ is well-defined
2. $\pi_n(G)$ is a group
3. When G is connected, $\pi_n(G)$ is independent of the base vertex v_0
4. $\pi_n(G)$ is abelian $\forall\, n \geq 2$
5. If X_G is the cell complex obtained by regarding G as a 1-complex and attaching a 2-cell along the boundary of each 3- and 4-cycle,

$\Rightarrow \quad \pi_1(G) \cong \pi_1(X_G)$

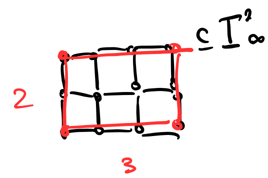
discrete fundamental group

topological fundamental group

§ Contractible graphs have trivial homotopy groups

Recall: A contraction of G is a homotopy from the identity map $id: G \to G$ to a constant map.

- A graph is contractible if it admits a contraction.
- finite trees are contractible (Dec 2 talk)
- grid graphs are contractible.



- $L \subseteq I_\infty^n$ a subgraph induced by the vertices $\{a_1, \dots, b_1\} \times \dots \times \{a_n, \dots, b_n\}$ where $a_i, b_i \in \mathbb{Z}$ and $a_i \leq b_i$ $\forall i$.
- The boundary of a grid graph, ∂L, is the subgraph induced by all vertices $x \in L$ where some component of x is equal to an a_i or b_i $\rightsquigarrow$ $x_i \in \{a_i, b_i\}$ for some i.
- The interior of a grid graph is the complement of the boundary, $L^\circ = L \setminus \partial L$.
- L is contractible:

Let $m := \max_{1 \leq i \leq n} (b_i - a_i)$, $\quad c : L \otimes I_m \to L$

$$c(v, i) := \left(\max\{v_1 - i, 0\}, \max\{v_2 - i, 0\}, \dots, \max\{v_n - i, 0\} \right)$$

- This is a graph map
- $c(-, 0) = id_L$
- $c(-, m) = 0$ (the zero map)

$(v_i - m \leq 0 \ \forall i.)$

$\boxed{\text{Proposition}}$ If G is contractible, then $\pi_n(G)$ is trivial for all n.

$\boxed{\text{Proof}}$ Let $v_0 \in V$ and $c: G \otimes I_m \to G$ a contraction of G to v_0.

• Recall: This means that c is a homotopy from id_G to the constant map at v_0 (sometimes denoted by v_0).

Let $f: (I_\infty^n, I_{\geq r}^n) \longrightarrow (G, v_0)$ be a graph map with radius r, so that $[f] \in \pi_n(G)$.

If there is a contraction c s.t. $c(v_0, -)$ is constant, say that G deformation retracts onto v_0.

Goal: $[f] = 0$ in $\pi_n(G)$

ie f is based homotopic to the constant v_0 map.

If the homotopy c is constant $\rightsquigarrow$ $c(v_0, i) = v_0 \;\; \forall i \in I_m$, then there is a based homotopy $H: I_\infty^n \otimes I_m \to G$ from f to v_0 defined by:

$$H(x,i) := c(f(x), i) \qquad \forall \;\; (x,i) \in I_\infty^n \otimes I_m.$$

• This is a graph map since c is.

• It is a based homotopy since c is constant, i.e,
$$H(v_0, i) = c(f(v_0), i) = c(v_0, i) = v_0 \quad \forall i$$

But this only works as $c(v_0, -)$ is the constant v_0 map.

In general, if $c(v_0, -)$ is not constant, there's no reason for $H(v_0, i) = v_0 \;\; \forall i$

$\rightsquigarrow$ ie, no guarantee of a based homotopy!

How do we fix this? Define a homotopy by defining one "half" of the homotopy at a time.

That is: define $h: I_\infty^n \otimes I_{2m} \to G$ by first defining h_i for $0 \leq i \leq m$, and then for $m+1 \leq i \leq 2m$.

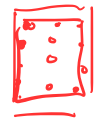
$0 \leq i \leq m$: FIRST HALF: Use the contraction c to stabilize the values inside the active area of f. The values outside the active area of f move one "boundary level" away at each time step (move it outwards)

$m+1 \leq i \leq 2m$: SECOND HALF: The map is constant on the active area of f, so all we have to do is move the dead area (constant portion) inwards one boundary level at a time to get a map that is constant everywhere.

More precisely...

FIRST HALF: We first define the idea of a "boundary level" more precisely.

Let $I^n_{\leq r} \subseteq I^n_\infty$ denote the grid graph induced by vertices



$$\underbrace{\{-r, -r+1, \dots, r-1, r\} \times \dots \times \{-r, -r+1, \dots, r-1, r\}}_{n \text{ sets}}$$

This r is the radius of f

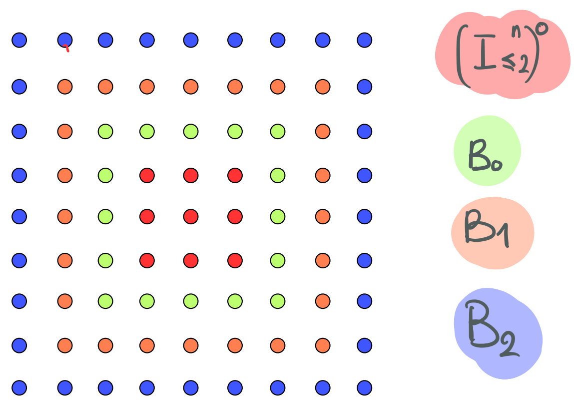
We can define a partition of I^n_∞ using this grid graph and some recursion:

$B_0 := \partial I^n_{\leq r}$ (bd of grid graph) $\rightsquigarrow$ denotes the boundary of the grid graph $I^n_{\leq r}$; x s.t. $x_i \in \{-r, r\}$ for some i.

For $K \geq 1$ $\rightsquigarrow$ $B_K := \partial G_{K-1}$ where G_{K-1} is the grid graph with interior all previous B_i's:

$$G^o_{K-1} := (I^n_{\leq r}) \cup B_1 \cup \dots \cup B_{K-1}$$

eg $r=2$, $n=2$



(notice that $B_0 \subseteq I^n_{\leq r}$)

h_0

$\rightsquigarrow$ $\{I^n_{\leq r}\} \cup \{B_K\}_{K \geq 1}$ forms a partition of I^n_∞.

These B_K's will keep track of the "boundary levels"

How do we define h_i for $i = 0, 1, \dots, m$?

$$h_0 = f$$

For $1 \leq i \leq m$:

$$h_i(x) = \begin{cases} c(f(x), i) & \text{if } x \in I^n_{\leq r} \\ h_{i-1}(x_0) & \text{if } x \in B_K, \text{ where } x_0 \in B_{K-1} \end{cases}$$

for $k \geq 1$

In words: The contraction tells us what to do inside the active area of f. On the dead area of f, take the value of h_{i-1} on the "smaller" boundary level.

$\Rightarrow h_m$ is constant on the active area of f $(I^n_{\leq r})$ and on each B_k.

<u>SECOND HALF</u>: Let $i \in \{m+1, m+2, \dots, 2m\}$,

$$h_i(x) = \begin{cases} h_m(x_0) & \text{if } x \in I^n_{\leq r} \cup B_1 \cup \dots \cup B_{i-m}, \text{ where } x_0 \in B_{i-m} \\ h_m(x) & \text{if } x \in B_k \text{ for } k > i-m. \end{cases}$$

In general, $h : I^n_\infty \otimes I_{2m} \to G$ is a based homotopy since it is defined recursively and at each time-step change, values only change according to what is adjacent.

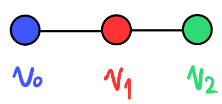
↳ In the first half: use the contraction which is also a graph map.

↳ It is a <u>based</u> homotopy because $\exists\, R > 0$ s.t.

$$h : (I^n_\infty \otimes I_m, I^n_{\geq R}) \to (G, v_0)$$

Let's try to see this with an example:

Let $G = I_2$ and $c : G \otimes I_4 \to G$ the contraction of G with the following values:

$G =$ 

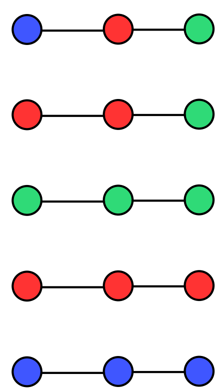
$c_0 = id_G$

c_1

c_2

c_3

$c_4 = const_{v_0}$

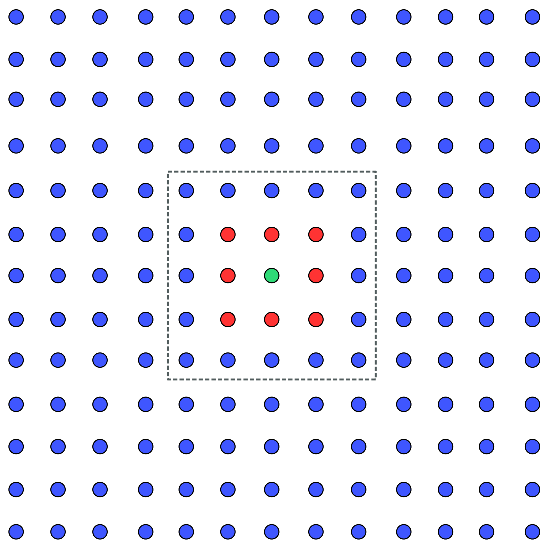


- Since c is defined in 4 time steps, $m = 4$.
- Represent a graph map $I^2_\infty \to G$ as an infinite 2-dimensional lattice of dots.
  ⇝ The colour of the dot $x \in I^2_\infty$ is the value of the map at x.

Let $f : (I^2_\infty, I^2_{\geq 2}) \to (G, v_0)$ be represented by the following array:

(when restricted to $I^2_{\leq 6}$)

$\boxed{f = h_0}$



- The dashed line represents the radius of f.
- Outside of $I^2_{\leq 6}$, the maps are constant at x_0 (ie, blue)
- The "first half" consists of maps h_0, h_1, h_2, h_3, h_4

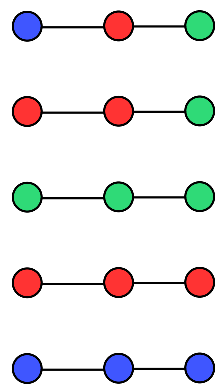
$c_0 = id_G$

c_1

c_2

c_3

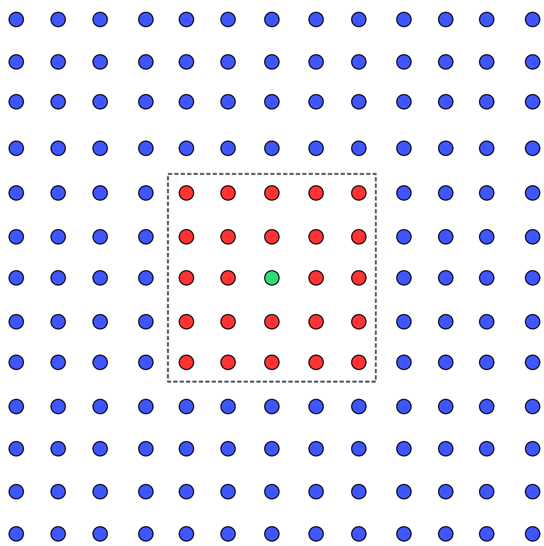
c_4



For $1 \leq i \leq 4$: $h_i(x) = \begin{cases} c(f(x), i) & \text{if } x \in I^2_{\leq 2} \\ h_{i-1}(x_0) & \text{if } x \in B_k, \text{ where } x_0 \in B_{k-1} \end{cases}$

for $k \geq 1$

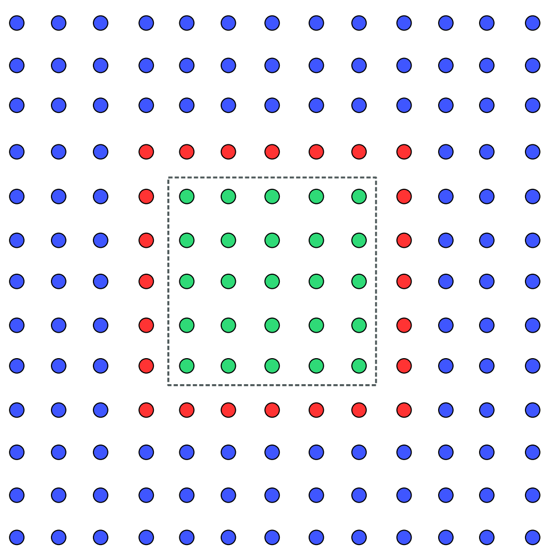
$\boxed{h_1}$



$$h_1(x) = \begin{cases} c(f(x), 1) & \text{if } x \in I^2_{\leq 2} \\ h_0(x_0) = f(x_0) & \text{if } x \in B_K,\ k \geq 1 \\ & \text{for } x_0 \in B_{k-1} \end{cases}$$

$\rightsquigarrow$ At time 1, $c(-, 1)$: blue $\mapsto$ red
red $\mapsto$ red
green $\mapsto$ green

$\boxed{h_2}$

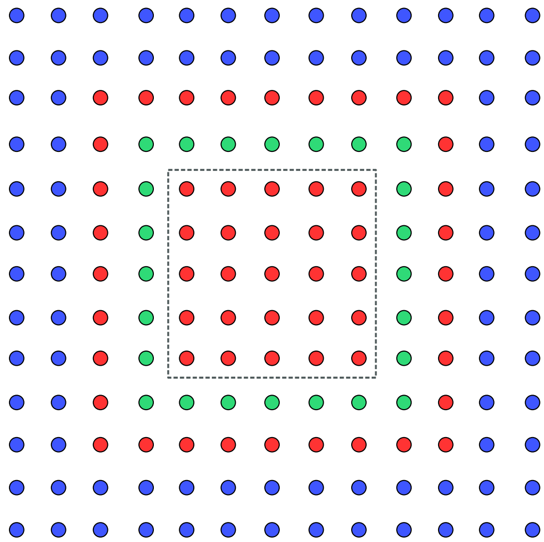


$$h_2(x) = \begin{cases} c(f(x), 2) & \text{if } x \in I^2_{\leq 2} \\ h_1(x_0) & \text{if } x \in B_K,\ k \geq 1 \\ & \text{for } x_0 \in B_{k-1} \end{cases}$$

$\rightsquigarrow$ At time 2, $c(-, 2)$: blue $\mapsto$ green
red $\mapsto$ green
green $\mapsto$ green

$\rightsquigarrow$ $h_2(x) = \underbrace{h_1(x_0)}_{= \text{RED}}$ if $x \in B_1$, $x_0 \in B_0$

$\boxed{h_3}$

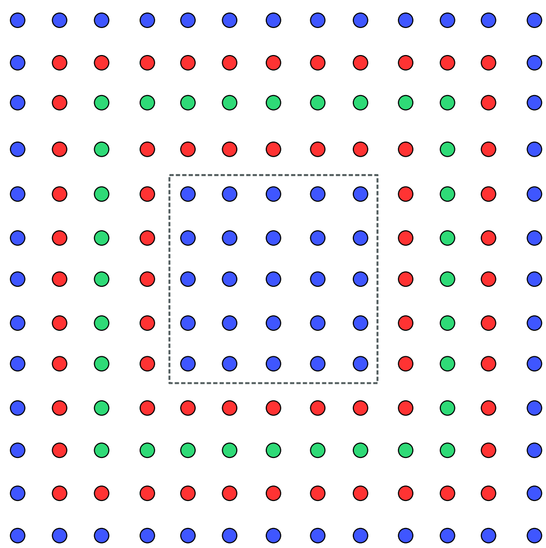


$$h_3(x) = \begin{cases} c(f(x), 3) & \text{if } x \in I^2_{\leq 2} \\ h_2(x_0) & \text{if } x \in B_k,\ k \geq 1,\ x_0 \in B_{k-1} \end{cases}$$

- $c(-, 3)$: B, G, R $\longmapsto$ red
- On B_1: $h_3(x) = h_2(x_0)$ where $x_0 \in B_0$
  $=$ GREEN
- On B_2: $h_3(x) = h_2(x_0)$ for $x_0 \in B_1$
  $=$ RED
- On B_3: $h_3(x) = h_2(x_0)$ for $x_0 \in B_2$
  $=$ BLUE

$\rightsquigarrow$ blue on all B_k, $k \geq 3$.

$\boxed{h_4}$



$$h_4(x) = \begin{cases} c(f(x), 4) & \text{if } x \in I^2_{\leq 2} \\ h_3(x_0) & \text{if } x \in B_k,\ k \geq 1, \text{ where } x_0 \in B_{k-1} \end{cases}$$

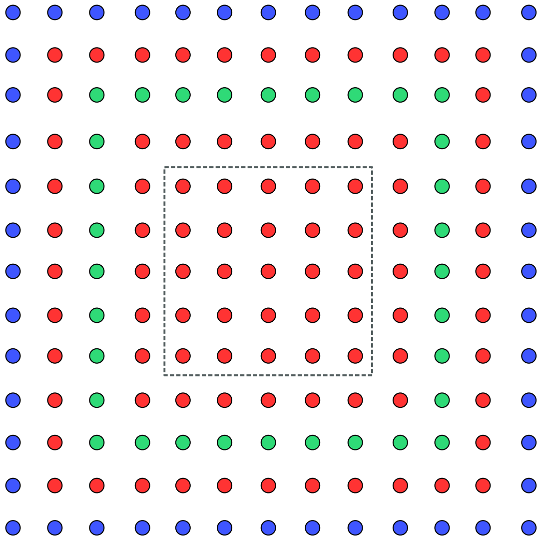
- $c(-, 4) =$ const v_0
- On B_1: $h_4(x) = h_3(x_0)$ for $x_0 \in B_0$
  $=$ RED
- On B_2: $h_4(x) = h_3(x_0)$ for $x_0 \in B_1$
  $=$ GREEN
- On B_3: RED

We move on to the "second half", $5 \leq i \leq 8$:

For $5 \leq i \leq 8$:

$$h_i(x) = \begin{cases} h_4(x_0) & \text{if } x \in I^2_{\leq 2} \cup B_1 \cup \cdots \cup B_{i-4}, \text{ where } x_0 \in B_{i-4} \\ h_4(x) & \text{if } x \in B_k \text{ for } k > i-4. \end{cases}$$

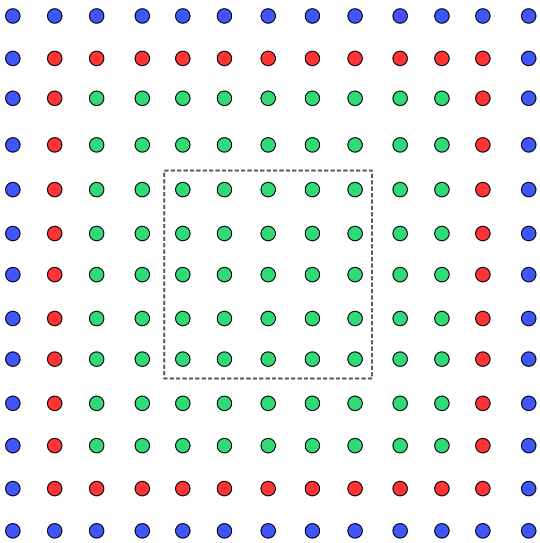
$\boxed{h_5}$



$$h_5(x) = \begin{cases} h_4(x_0) & \text{if } x \in I^2_{\leq 2} \cup B_1,\ x_0 \in B_1 \\ h_4(x) & \text{if } x \in B_k \text{ for } k > 1 \end{cases}$$

$\underline{\text{On } I^2_{\leq 2} \cup B_1:}$ $h_5(x) = \text{RED}$

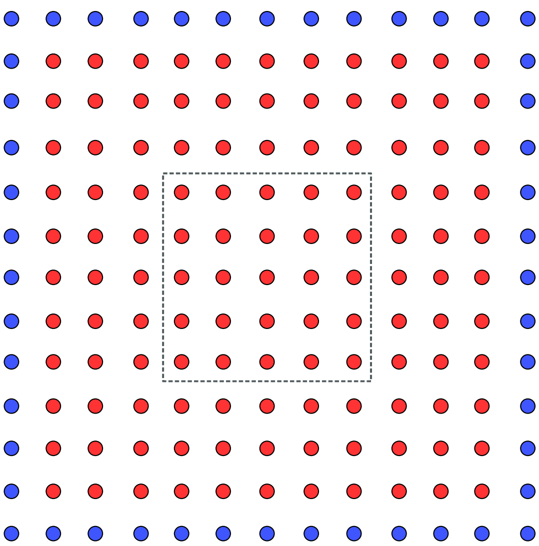
$\boxed{h_6}$



$$h_6(x) = \begin{cases} h_4(x_0) & \text{if } x \in I^2_{\leq 2} \cup B_1 \cup B_2,\ x_0 \in B_2 \\ h_4(x) & \text{if } x \in B_k \text{ for } k > 2 \end{cases}$$

$\underline{\text{On } I^2_{\leq 2} \cup B_1 \cup B_2:}$ $h_6(x) = \text{GREEN}$

$\boxed{h_7}$

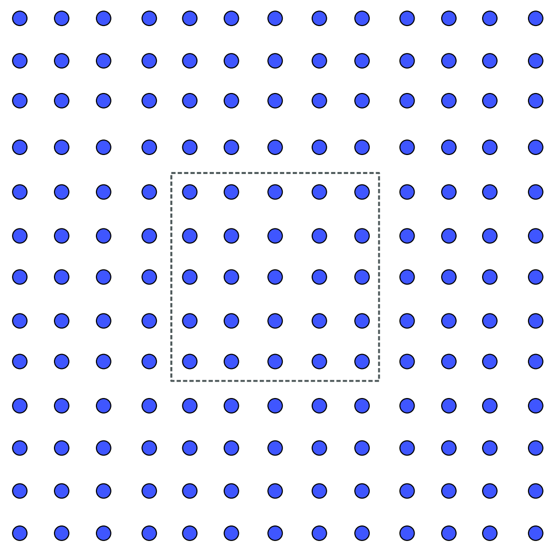


$$h_7(x) = \begin{cases} h_4(x_0) & \text{if } x \in I^2_{\leq 2} \cup B_1 \cup B_2 \cup B_3,\ x_0 \in B_3 \\ h_4(x) & \text{if } x \in B_k \text{ for } k > 3 \end{cases}$$

$\underline{\text{On } I^2_{\leq 2} \cup B_1 \cup B_2 \cup B_3:}$

$h_7(x) = \text{RED}$

$\boxed{h_8}$



$$h_8(x) = \begin{cases} h_4(x_0) & \text{if } x \in I^2_{\le 2} \cup B_1 \cup B_2 \cup B_3 \cup B_4 \\ & \quad x_0 \in B_4 \\ h_4(x) & \text{if } x \in B_k \text{ for } k > 4 \end{cases}$$

On $I^2_{\le 2} \cup B_1 \cup B_2 \cup B_3 \cup B_4$:

$h_8(x) =$ BLUE

That is, $[f] = [0]$

$$\Rightarrow \pi_n(G) = \{0\}$$

□

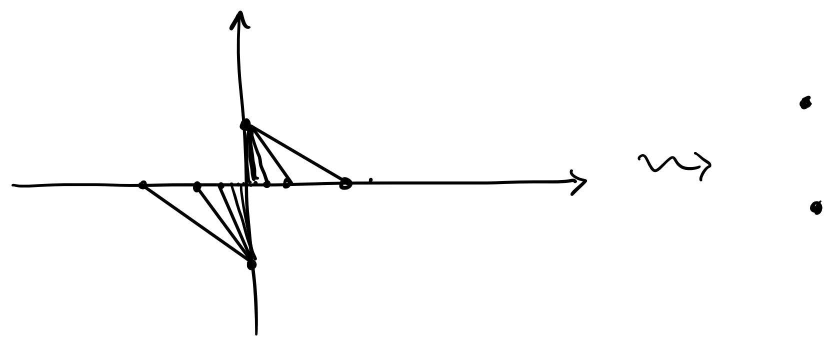
Note: The reason why h_8 was constant in the above example is the reason why h_{2m} is constant in general ⇝ we moved things into the "dead area" using the m steps of the contraction, so after another m steps, we reach "dead area" again.

Q: Suppose G is contractible. Is there a deformation retract of G onto some $v_0 \in G$?

True?

- Simplicial / CW approximation?

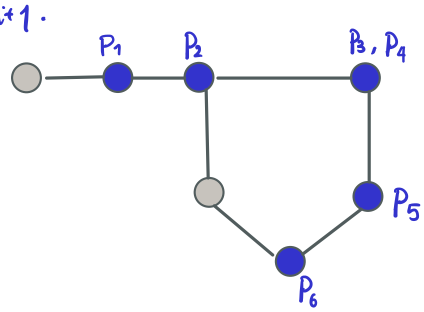
contractible but not a deformation retract:



- Are graphs CW complexes in this setting?

§ The homotopy groups of graphs with no 3- or 4-cycles

$\boxed{\text{Def}}$ A path in G is a sequence $(p_1, p_2, \ldots, p_k)$ of vertices of G s.t. for all i, either $p_i = p_{i+1}$ or $p_i \sim p_{i+1}$.



$\boxed{\text{Theorem}}$ If G contains no 3- or 4-cycles, then $\pi_n(G)$ is trivial $\forall\, n \geq 2$.

- Recall that if G has no 3- or 4-cycles, then $\pi_1(G)$ is the same as the fundamental group of G as a topological space

  ⇝ Theorem determines the remaining homotopy groups.

$\boxed{\text{Proof}}$ · G has no 3- or 4-cycles

- $n \geq 2$, $v_0 \in V$
- $f : (I_\infty^n, I_{\geq r}^n) \to (G, v_0)$ graph map of radius $r > 0$.

Goal: $[f] = [v_0]$ in $\pi_n(G)$.

Ingredients:

- For this proof, consider G as a DIRECTED GRAPH with edges E so that

$$(u,v) \in E \Rightarrow (v,u) \notin E$$

(Add a direction to each edge)

- $F_E :=$ free group on E
  - Identity: 1
  - Elements: reduced words in the letters e and e^{-1} $\forall\, e \in E$

- $\mathcal{P}_2(I_\infty^n) :=$ set of paths in I_∞^n of length at least 2

- Define a function $\tau : P_2(I^n_\infty) \longrightarrow F_E$:
  - For adjacent vertices $x, y \in I^n_\infty$:

$$\tau(x,y) := \begin{cases} (f(x), f(y)) & \text{if } (f(x), f(y)) \in E \\ (f(y), f(x))^{-1} & \text{if } (f(y), f(x)) \in E \\ 1 & \text{if } f(x) = f(y) \end{cases}$$

Since f is a graph map, one of these must hold so it is a complete definition

  - For a path $P = (p_0, p_1, \dots, p_\ell)$ in I^n_∞, $\ell \geq 1$, define:

$$\tau(P) := \tau(p_0, p_1) \cdot \tau(p_1, p_2) \cdot \ \cdots \ \cdot \tau(p_{\ell-1}, p_\ell)$$

This is concatenation of "words" in F_E

In words: τ takes a path in I^n_∞ and traces out all of its edges (according to what f does) as "reduced words" in F_E ⇝ tells you which edges are traversed along the path (keeping the directed edges in mind)

- Define $g : I^n_\infty \to F_E$ by

$$g(x) := \tau(P)$$

where P is any path in I^n_∞ starting in $I^n_{\geq r}$ ("dead area" of f) and ending at x

"Any path" ⇝ Is g well-defined?

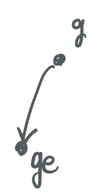
Recipe:
1. Verify that g is well-defined.
2. Use some machinery (subtree of a Cayley graph) to show that $[f] = [v_0]$

Suppose 1 holds.

Let $\Gamma(F_E, E)$ denote the Cayley graph of F_E with generating set E

(ie) $V(\Gamma(F_E, E)) := F_E$

For each $e \in E \cup E^{-1}$, $g \in F_E$, draw an edge (g, ge)



Let $G_E \subseteq \Gamma(F_E, E)$ be the subgraph induced by $\mathrm{Im}(\tau)|_{I^n_{\geq r}}$.

$$V_E := \{ \tau(P) \mid P \text{ is a path in } I^n_\infty \text{ starting in } I^n_{\geq r} \}$$

paths starting in the "dead area" of f.

vertices of the induced subgraph.

Notice: G_E is finite (since f has finite support) and it has no cycles since $\Gamma(F_E, E)$ doesn't.

General fact about Cayley graphs: $\Gamma(H,S)$ is a tree if H is a free group with generating set S.

Each of the edges of $\Gamma(H,S)$ corresponds to multiplication by one of the generators $s \in S \rightsquigarrow$ get one directed edge.

If you multiply by the inverse s^{-1}, you get the other directed edge.

If you have a cycle in $\Gamma(H,S)$, then you'll have

$$e \cdot g_1 \cdot g_2 \cdots g_n = e$$

$$\Rightarrow g_1 \cdot g_2 \cdots g_n = 1$$ ↯ contradicts free group axioms. (free group has no relations)

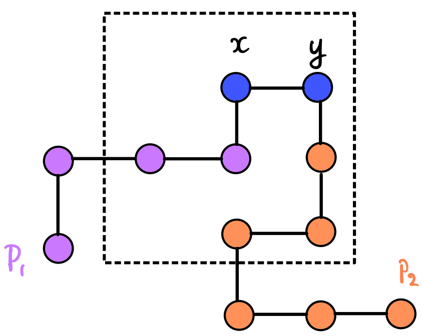
Now, the distance between $u, v \in V_E$ is $|u^{-1}v|$ where this is the length of the reduced word representing $u^{-1}v \in F_E$.

$g : I^n_\infty \to G_E$

Suppose either $x = y$ or $x \sim y$ in I^n_∞. Then,

$|(f(x), f(y))|$, $|(f(x), f(y))^{-1}|$ or 1

$$|g(x)^{-1} g(y)| = |\tau(x,y)|$$



[WHY?

$g(y) = \tau(P_2)$

$g(y) = \tau(P_1) \cdot \tau(x,y)$ (as g is well-defined)

$g(y) = g(x) \cdot \tau(x,y)$

$\Rightarrow |g(x)^{-1} g(y)| = |\tau(x,y)|$ ⅃

no contribution to the length in the dead area of f since f is constant here

So the lengths of the reduced words are the same.

Since $|\tau(x,y)| \leq 1$, then

$\Rightarrow |g(x)^{-1} g(y)| \leq 1$

That is, $g(x) = g(y)$ or $g(x) \sim g(y)$ in G_E.

[Which words have length 1 or 0?]

Therefore, $g : (I^n_\infty, I^n_{\geq r}) \longrightarrow (G_E, 1)$ is a graph map.

(adjacent vertices are either mapped to the same vertex or to two adjacent vertices)

Since G_E is a finite tree, it is contractible.

By preceding proposition, then $\pi_n(G_E) = \{0\}$ $\forall$ n.

As $[g] \in \pi_n(G_E)$, then there exists a based homotopy $h: (I^n_\infty \otimes I_m, I^n_{\geq R}) \to (G_E, 1)$

from g to the constant map 1:

$$\begin{cases} h_0 = g \\ h_m = 1 \end{cases}$$

from the "dead area" of f

Now, every vertex of G_E corresponds to a path in G from v_0 to some vertex,

denote this vertex by $\pi(v)$

Then, $\pi : (G_E, 1) \to (G, v_0)$

$$v \longmapsto \pi(v)$$

is a graph map.

Also, notice that

$$\pi \circ g = f \qquad \text{by definition of } g(x) := \tau(P)$$

$\Rightarrow$ $\pi \circ h : (I^n_\infty \otimes I_m, I^n_{\geq R}) \longrightarrow (G, v_0)$ is a based homotopy where

$$(\pi \circ h)_i : (I^n_\infty, I^n_{\geq R}) \to (G, v_0) \quad \text{and} \quad (\pi \circ h)_0 = \pi \circ h_0 = \pi \circ g = f$$

$$(\pi \circ h)_m = \pi \circ h_m = \pi \circ 1 = v_0$$

constant map at v_0

That is, $[f] = [v_0]$

$\Rightarrow$ $\pi_n(G)$ is trivial.

Since (2) holds, it remains to show that (1) holds: g is well-defined

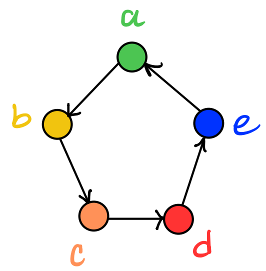
where $g: I^n_\infty \to F_E$ is defined by

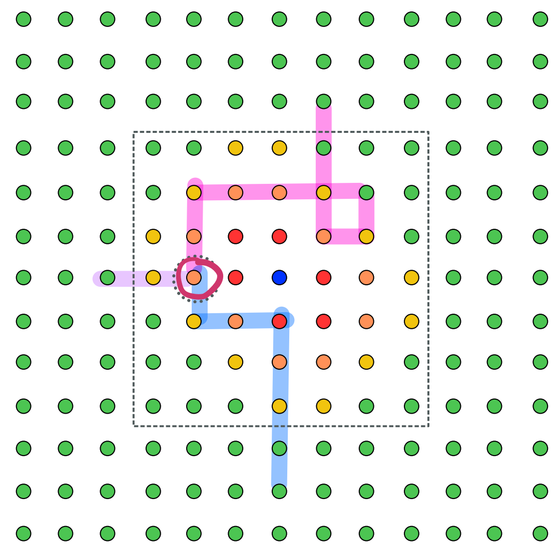
$$g(x) := \tau(P)$$

where P is any path in I^n_∞ starting in $I^n_{\geq r}$ ("dead area" of f) and ending at x

Let's look at an example first...

$G = C_5$, $f : (I^2_\infty, I^2_{\geq 4}) \to (C_5, a)$ given by:

$G =$ 

$f =$ 

Let $x = (0,-2)$

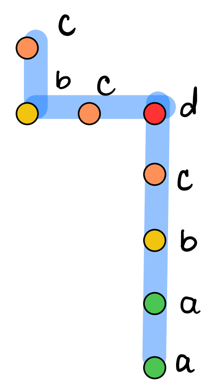
$\leadsto f(x) = c$ (orange)

Three highlighted paths give 3 ways to compute $g(x)$ $\leadsto$ they all give the same value:

- Purple path: $g(x) = (a,b)(b,c)$

- Blue path: $g(x) = (a,b)(b,c)(c,d)(c,d)^{-1}(b,c)^{-1}(b,c)$

$= (a,b)(b,c)$



Recall that for $x \sim y$, if $(f(y), f(x)) \in E$
$T(x,y) = (f(y), f(x))^{-1}$

- Pink path: $g(x) = (a,b)(b,c)(b,c)^{-1}(b,a)^{-1}(a,b)(b,c)(c,c)(b,c)^{-1}(b,c)(c,c)$

$= (a,b)(b,c)$

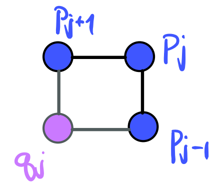
So the definition of g (using f) forces it to be well-defined in this case, and it does so in the general case as well.

$\leadsto$ This is where we use that G has no 3- or 4-cycles.

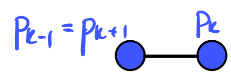
To show that g is well-defined, let $P = (p_0, \dots, p_\ell)$ be a path in I^n_∞.

We define two operations on P:

- If for some $0 < j < \ell$, the vertices p_{j-1}, p_j, p_{j+1} are the three corners of a square in I^n_∞, let q_j denote the furth corner.
  - If we replace p_j in P with q_j, this is called a corner swap.



- If $p_{k-1} = p_{k+1}$ for some $0 < k < \ell$, if we remove the terms p_{k-1} and p_k from P, this is called a backtrack deletion.



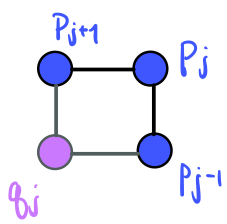
<u>Claim 1</u>: $\tau(P)$ is invariant under both operations (corner swaps, backtrack deletions)

<u>Proof of claim</u>:

- <u>Corner swaps</u>:

  Let p_{j-1}, p_j, p_{j+1} be three corners of a square in I^n_∞ and q_j the furth corner

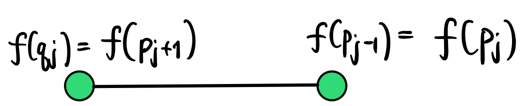
  Let S denote the square.

$S =$ 

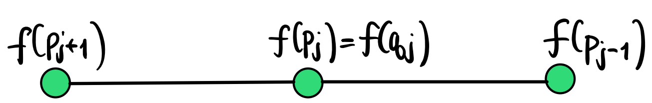
Since f is a graph map and G does not have any 4-cycles (i.e., it has no squares)

$\Rightarrow$ $f(S)$ is a path in G with at <u>MOST</u> 3 vertices. (a <u>path</u> since G has no 3-cycles)

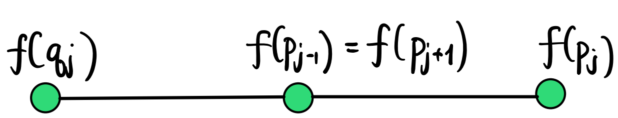
The nontrivial possibilities are:

<u>CASE I</u>: 

contract the two vertical edges

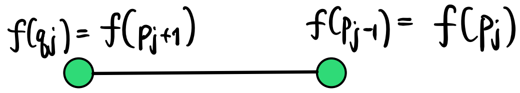
<u>CASE II</u>: 

contract the two edges adjacent to q_j

<u>CASE III</u>: 

contract the two edges adjacent to p_{j-1}.

CASE I:



- For adjacent vertices $x, y \in I^n_\infty$:

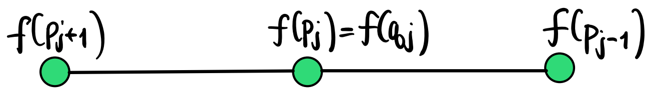
$$\tau(x,y) := \begin{cases} (f(x), f(y)) & \text{if } (f(x), f(y)) \in E \\ (f(y), f(x))^{-1} & \text{if } (f(y), f(x)) \in E \\ 1 & \text{if } f(x) = f(y) \end{cases}$$

By definition of τ, then

$$\begin{aligned} \tau(p_{j-1}, p_j, p_{j+1}) &= \tau(p_{j-1}, p_j) \cdot \tau(p_j, p_{j+1}) \\ &= 1 \cdot \tau(p_j, p_{j+1}) \\ &= \tau(p_j, q_j) \\ &= \tau(p_j, q_j) \cdot \tau(q_j, p_{j+1}) \\ &= \tau(p_j, q_j, p_{j+1}) \end{aligned} \quad \Big\} \text{ since } f(q_j) = f(p_{j+1})$$

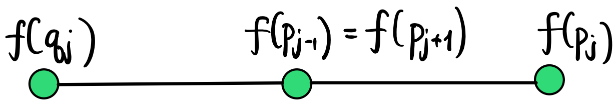
$$\rightsquigarrow \tau(p_{j-1}, p_j, p_{j+1}) = \tau(p_{j-1}, q_j, p_{j+1}) \qquad \text{since } f(p_j) = f(p_{j-1})$$

CASE II:



$$\begin{aligned} \tau(p_{j-1}, p_j, p_{j+1}) &= \tau(p_{j-1}, p_j) \cdot \tau(p_j, p_{j+1}) \\ &= \tau(p_{j-1}, q_j) \cdot \tau(q_j, p_{j+1}) \\ &= \tau(p_{j-1}, q_j, p_{j+1}) \end{aligned}$$

CASE III:



$$\begin{aligned} \tau(p_{j-1}, p_j, p_{j+1}) &= \tau(p_{j-1}, p_j) \cdot \tau(p_j, p_{j+1}) \\ &= \tau(p_{j-1}, p_j) \cdot \tau(p_j, p_{j-1}) \\ &= \tau(p_{j-1}, q_j, p_{j+1}) \end{aligned}$$

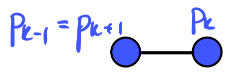
In all cases, $\tau(P) = \tau(p_0, \dots, p_{j-1}, p_j, p_{j+1}, \dots, p_\ell)$

$$\begin{aligned} &= \tau(p_0, \dots, p_{j-1}) \cdot \tau(p_{j-1}, p_j, p_{j+1}) \cdot \tau(p_{j+1}, \dots, p_\ell) \\ &= \tau(p_0, \dots, p_{j-1}) \cdot \tau(p_{j-1}, q_j, p_{j+1}) \cdot \tau(p_{j+1}, \dots, p_\ell) \end{aligned}$$

$$= \tau(p_0, p_1, \dots, p_{j-1}, q_j, p_{j+1}, \dots, p_\ell)$$

$\Rightarrow \tau(P)$ is invariant under corner swaps.

- <u>Backtrack deletions:</u> Suppose $p_{k-1} = p_{k+1}$ for some $k < \ell$.



Then, $\tau(p_{k-1}, p_k, p_{k+1}) = \tau(p_{k-1}, p_k) \cdot \tau(p_k, p_{k+1})$

$= 1$

$$\rightsquigarrow \tau(P) = \tau(p_0, \dots, p_{k-1}, p_k, p_{k+1}, \dots, p_\ell)$$

$$= \tau(p_0, \dots, p_{k-1}) \cdot \underbrace{\tau(p_{k-1}, p_k, p_{k+1})}_{1} \cdot \tau(p_{k+1}, \dots, p_\ell)$$

$$= \tau(p_0, \dots, p_{k-1}) \cdot \tau(p_{k+1}, \dots, p_\ell)$$

$$= \tau(p_0, \dots, p_{k-2}, p_{k+1}) \cdot \tau(p_{k+1}, \dots, p_\ell) \qquad \text{as } p_{k-1} = p_{k+1}$$

$$\Rightarrow \tau(P) = \tau(p_0, \dots, p_{k-2}, p_{k+1}, \dots, p_\ell)$$

$\hookrightarrow$ This is exactly the path P with the backtrack deleted.

$\therefore \tau(P)$ is invariant under corner swaps and backtrack deletions, so the claim holds. ∎

<u>Claim 2</u>: If $P = (p_0, \dots, p_\ell)$ is a <u>closed</u> path in I^n_∞, then $\tau(P) = 1$

<u>Proof of claim</u>: Any closed path can be transformed into the path (p_0, p_1, p_0) by a sequence of corner swaps and backtrack deletions.

By <u>claim 1</u>, then

$$\tau(P) = \tau(p_0, p_1, p_0)$$

$$= \tau(p_0, p_1) \cdot \tau(p_1, p_0)$$

$$= 1$$

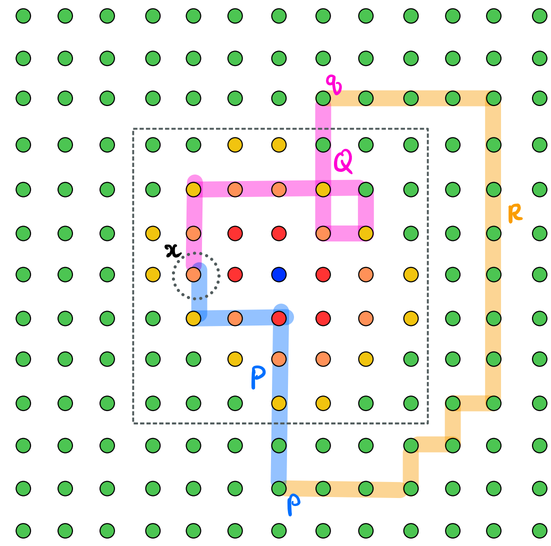
∎

Finally, we are ready to show that g is well-defined.

Let $x \in I^n_\infty$ and $p, q \in I^n_{\geq r}$.

Let P be a path from p to x.

Let Q be a path from p to x.

$f =$ 

Want to show: $\tau(P) = \tau(Q)$.

To this end, let R be a path from q to p with all vertices in $I^n_{\geq r}$.

Since f is constant on $I^n_{\geq r}$, it is constant on R and so

$$\tau(R) = 1.$$

Let us traverse along P, reverse along Q, and then along R.

Denote this path by $PQ^{-1}R$. This is a closed path, so by Claim 2,

$$\tau(PQ^{-1}R) = 1$$

$$\Rightarrow 1 = \tau(PQ^{-1}R)$$

$$= \tau(P) \cdot \tau(Q)^{-1} \cdot \tau(R)$$

$$\Rightarrow 1 = \tau(P) \cdot \tau(Q)^{-1}$$

$$\Rightarrow \tau(P) = \tau(Q)$$

Thus, g is well-defined and that completes the proof of the theorem. $\square$

§ Discrete Hurewicz

- A notion of "discrete singular cubical homology group" of a graph G, $H_n(G)$

$\boxed{\text{Dimension 1 Hurewicz Theorem}}$ For any graph G, there is a surjective map

$$\varphi : \pi_1(G) \longrightarrow H_1(G)$$

with $\operatorname{Ker}\varphi = [\pi_1(G), \pi_1(G)]$

BARCELO, CAPRARO, WHITE "Discrete homology theory for metric spaces"

$\boxed{\text{Theorem}}$ There is an infinite family of graphs $\{G\}$ such that $\varphi : \Pi_n(G) \longrightarrow H_n(G)$ is surjective.

LUTZ: Thm 5.10

- obtained by applying the "suspension" functor successively.
- If the first graph has no 3- or 4-cycles, the map is surjective.

Q: Is the map injective for certain graphs?