10.11.2021

① Definitions

② Homotopy invariance — Thm 3.4 in the first paper.

③ Computational techniques — disjoint unions, etc.

④ $H_d^{cube}(G) = 0 \quad \forall d \geq 2$ when G has no 3- or 4-cycles. — vanishing paper

Briefly.

An introduction to discrete cubical homology

- Today we'll be talking about a discrete homology theory on graphs.
- This was developed after discrete homotopy theory on undirected graphs that we've dealt with.
- There are more general homology theories, such as one on metric spaces and another on directed graphs.
- We will focus on the so-called cubical homology theory on graphs.
- We will see differences between this framework and the classical singular (/simplicial) homology on graphs (viewing them as 1-dim complexes)
- In particular, the 4-cycle has trivial cubical homology in all dims > 0.

① Definitions and setup

Notation:

- R: commutative ring with unity. (will be the ring of coefficients for the cubical homology theory)
- $n \in \mathbb{N}_{\geq 1} \rightsquigarrow [n] := \{1, 2, 3, .., n\}$

$\boxed{\text{def}}$ Let $n \geq 1$.

- The DISCRETE n-CUBE (graph) Q_n is the graph with:

$$V(Q_n) := \{0,1\}^n = \{(a_1, a_2, .., a_n) : a_i \in \{0,1\} \ \forall i \in [n]\}$$

$$E(Q_n) := \{\{a, b\} : \exists\, i \in [n] \text{ s.t. } a_i \neq b_i \text{ and } a_j = b_j \ \forall j \neq i\}$$

ie They differ in exactly one position.
"Hamming distance is 1"

- Let $Q_0 := \bullet$

- G, H simple graphs $\rightsquigarrow$ graph map $\sigma: G \to H$ is a map on vertices that either maps edges to edges (preserving vertex adjacency) or contracts the edge to a single vertex.
- A graph map $\sigma: Q_n \to G$ is called a SINGULAR n-CUBE on G.

In singular/simplicial homology, we take a continuous function $\Delta^n \to X$

standard n-simplex

topological space

- For each $n \geq 0 \rightsquigarrow \mathcal{L}_n^{Cube}(G) := \langle \sigma : Q_n \to G \mid \sigma \text{ is a graph map}, n \geq 0 \rangle_{R\text{-mod}}$: the free R-module generated by all singular n-cubes. (so consists of formal sums)

$\odot \to G$

- $n \geq 1,\ i \in [n] \xrightarrow[\text{face maps}]{\text{define}} f_i^+, f_i^- : \mathcal{L}_n^{Cube}(G) \longrightarrow \mathcal{L}_{n-1}^{Cube}(G)$ (from n-cubes to (n-1)-cubes)

$\sigma : Q_n \to G$ so takes n arguments.

$\sigma \in \mathcal{L}_n^{Cube}(G),\ (a_1, .., a_{n-1}) \in Q_{n-1} \rightsquigarrow$ $f_i^+ \sigma(a_1, a_2, .., a_{n-1}) := \sigma(a_1, a_2, .., a_{i-1}, 1, a_i, .., a_{n-1})$

$f_i^- \sigma(a_1, a_2, .., a_{n-1}) := \sigma(a_1, a_2, .., a_{i-1}, 0, a_i, .., a_{n-1})$

- $\sigma : Q_n \to G$ is called DEGENERATE if $f_i^+ \sigma = f_i^- \sigma$ for some $i \in [n]$. Otherwise, σ is NON-DEGENERATE
- def: any 0-cube $\bullet \xrightarrow{\sigma} G$ is non-degenerate.

- $n \geq 0 \rightsquigarrow D_n^{Cube}(G) := \langle \sigma \in \mathcal{L}_n^{Cube}(G) \mid \sigma \text{ is degenerate} \rangle \leq \mathcal{L}_n^{Cube}(G)$ submodule

$\rightsquigarrow \mathcal{C}_n^{Cube}(G) := \mathcal{L}_n^{Cube}(G) / D_n^{Cube}(G)$ (free R-module)

- Elements of $\mathcal{C}_n^{Cube}(G)$ are called n-CHAINS

Claim: $\{\mathcal{C}_n^{Cube}(G)\}_{n \geq 0}$ forms a chain complex of free R-modules under the following boundary operator:

$n \geq 1 \rightsquigarrow \partial_n^{Cube} : \mathcal{L}_n^{Cube}(G) \to \mathcal{L}_{n-1}^{Cube}(G)$ s.t.

$\sigma \in \mathcal{L}_n^{Cube}(G) \rightsquigarrow \partial_n^{Cube}(\sigma) := \sum_{i=1}^{n} (-1)^i (f_i^- \sigma - f_i^+ \sigma)$

and extend it linearly to all n-chains in $\mathcal{L}_n^{Cube}(G)$

So this is a graph map $Q_{n-1} \to G$

- Also, $\mathcal{L}_{-1}^{Cube}(G) := D_{-1}^{Cube}(G) = (0)$, define $\partial_0^{Cube} : \mathcal{L}_0^{Cube}(G) \to \mathcal{L}_{-1}^{Cube}(G) = (0)$ as the trivial map.

To check: $\bullet\ \partial_n [D_n^{Cube}(G)] \subseteq D_{n-1}^{Cube}(G)$

(i.e., it's a chain complex!) $\bullet\ \partial_n \partial_{n+1} \sigma = 0$

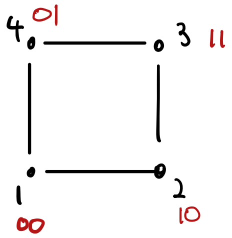
$\to \mathcal{C}_n \to \mathcal{C}_{n-1} \to \cdots \mathcal{C}_0$

- As usual: represent each coset in $\mathcal{C}_n^{Cube}(G)$ by the unique coset representative s.t. all forms are non-degenerate.

• Now we are ready to define the discrete cubical homology group — it's what you'd expect — the singular homology of the chain complex that we just described.

def: For $n \geq 0$, $\mathcal{H}_n^{Cube}(G) := {}^{\operatorname{Ker}\partial_n}/_{\operatorname{Im}\partial_{n+1}}$ is the n^{th} homology group of the chain complex $C_\bullet^{Cube}(G)$.
These are the <u>discrete cubical homology groups of the graph G.</u>

eg $G = C_4$



$C_0 = \langle (1), (2), (3), (4) \rangle$ all cubes $\bullet \longrightarrow G$, $\bullet \longrightarrow V(G)$

$C_1 = \langle (1,2), (2,1), (2,3), (3,2), \ldots (4,1), (1,4) \rangle$

$C_2 = \langle (1,1,1,2), (1,1,1,4), \ldots, (4,4,4,3) \rangle$

• rank $\partial_1 = |V| - 1 = 3$
the matrix is the vertex-directed edge incidence matrix of the corresponding directed graph.

• nullity $\partial_1 = 8 - 3 = 5$

• cycles in $C_1 \mapsto$ CIRCULATIONS in G (weighted sums of edges in which the net flow out of each vertex equals 0).

- basis of C_1: any directed cycle basis

eg
$$\left.\begin{array}{l} (1,2) + (2,1) \\ (2,3) + (3,2) \\ (3,4) + (4,3) \\ (4,1) + (1,4) \\ (1,2) + (2,3) + (3,4) + (4,1) \end{array}\right\} \text{each is a boundary of a 2-chain.}$$

Represent singular n-cubes $\sigma: Q_n \to G$ by sequences $(c_1, c_2, \ldots, c_{2^n})$ of length 2^n where $c_i := \sigma(i)$
the i^{th} term is the value of σ on the "i^{th}" vertex of Q_n and vertices of Q_n are ordered in colexicographic order.

$\rightsquigarrow H_0(G) = \mathbb{R}$
$H_1(G) = (0)$

Actually ... $H_n(G) = (0)$ $\forall$ $n > 0$
But there's a better way to compute them!

② Homotopy invariance

Recall:

• The BOX PRODUCT of graphs, $G \otimes H := \begin{cases} V(G) \times V(H) \\ \{(g_1,h_1),(g_2,h_2)\} \text{ s.t. } g_1 = g_2 \text{ and } h_1 \sim h_2 \text{ or } h_1 = h_2 \text{ and } g_1 \sim g_2 \end{cases}$

• $f, g: G \to H$ are homotopic if $\exists$ $\Phi: G \otimes I_m \to H$
s.t. $\Phi(-,0) = f$
$\Phi(-,m) = g$

• $G \simeq H$ (homotopy equivalent) if $\exists$ $f: G \to H$, $g: H \to G$ s.t. $gf \simeq id_G$ and $fg \simeq id_H$

A key computational tool

$\boxed{\text{Theorem}}$ If $G \simeq_h H$, then $\forall n \geq 0$, $H_n(G) \cong H_n(H)$

$\boxed{\text{proof}}$ To show: If $\alpha, \beta: G \to H$ are homotopic $\Rightarrow$ induce identical maps on homology.

Suppose $\Phi: G \otimes I_m \to H$ is a homotopy from α to β with

$$x \in V(G) \rightsquigarrow \begin{array}{l} \Phi(x,0) = \alpha(x) \\ \Phi(x, m) = \beta(x) \end{array}$$

For $\sigma \in C_n(G)$, define $\Phi(\sigma, j)(q) := \Phi(\sigma(q), j)$ $\forall q \in Q_n$, and a fixed j.

Define $\tilde{\alpha}_n(\sigma) := \Phi(\sigma, 0)$

$\tilde{\beta}_n(\sigma) := \Phi(\sigma, m)$

Fact: $\tilde{\alpha}$ and $\tilde{\beta}$ are chain maps ($\tilde{\alpha}_{n-1}\partial_n = \partial_n \tilde{\alpha}_n$ for both α, β)

Construct: chain homotopy between $\{\tilde{\alpha}_n\}_n$ and $\{\tilde{\beta}_n\}_n$.

$h_n: C_n(G) \longrightarrow C_{n+1}(H)$ s.t.

$\tilde{\beta}_n - \tilde{\alpha}_n = \partial_{n+1} h_n + h_{n-1}\partial_n$ $\forall n$.

construct using $\Phi, f_i^{\pm}, \ldots$

Once you have this, suppose $z \in C_n(G)$. Then,

$$\tilde{\beta}_n(z) - \tilde{\alpha}_n(z) = \partial_{n+1} h_n(z)$$

$$\Rightarrow \tilde{\beta}_n(z) - \tilde{\alpha}_n(z) \in \operatorname{Im}(\partial_{n+1})$$

$\Rightarrow$ same homology class $\forall z$

$\Rightarrow$ α and β induce the same maps on homology.

• In particular:

$\sigma \in C_n(G)$

$j = 1, 2, \ldots, m$.

• $h_n^{(j)}(\sigma) \in C_{n+1}(H)$ is the unique labelled $(n+1)$-cube s.t.

$$f_1^+ h_n^{(j)}(\sigma)(q) = \Phi(\sigma(q), j)$$
$$f_1^- h_n^{(j)}(\sigma)(q) = \Phi(\sigma(q), j-1) \quad \forall q \in Q_n$$

define $\rightsquigarrow$ $h_n(\sigma) := h_n^{(1)}(\sigma) + h_n^{(2)}(\sigma) + \cdots + h_n^{(m)}(\sigma)$

and then use def^{ns} to check the equality.

□

③ Computations

- So now let's move onto actually computing the homology groups of a graph.
- Some of the computations involve deformation retractions and the homotopy invariance fact – similar to what we do when trying to do computations for spaces.

def: G a graph.

$H \subseteq G$ an induced subgraph $\rightsquigarrow$ $V(H) \subseteq V(G)$

$E(H) := \{e \in E(G) \mid \text{both endpoints are in } V(H)\}$.

- A RETRACTION of G ONTO H is a graph map $r: G \to H$ s.t. $r(h) = h \ \forall \ h \in V(H)$
- A DEFORMATION RETRACTION of G onto H is a retraction $r: G \to H$ s.t. $ir \simeq_h id_G$ where $i: H \hookrightarrow G$ is the inclusion map
- A ONE-STEP DEFORMATION RETRACTION from G to H is a def. retr^n r s.t. $m=1$ in the homotopy $ir \simeq_h id_G$.

(ie) r is a retraction such that $\{x, r(x)\}$ is an edge $\forall \ x \in V(G)$.

- Now, we have the following lemma by homotopy invariance since $ri = id_H$.

Lemma If r is a deformation retraction from G onto a subgraph H, $i: H \hookrightarrow G$, then $G \simeq_h H$ (by r and i) and so

$$\mathcal{H}_n(G) \cong \mathcal{H}_n(H) \qquad \forall \ n \geq 0$$

- We get even more from this, that there are many infinite classes of graphs that have trivial homology.

Corollary If G is a tree, a complete graph, or a hypercube, then $\mathcal{H}_n(G) = (0) \ \forall \ n > 0$.

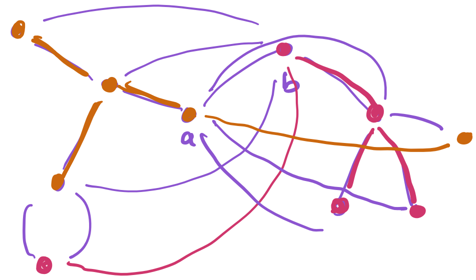
Proof
- tree : contractible ✓
- complete graph : contractible. ✓
- hypercube : define r by collapsing any facet onto its op. facet

$\rightsquigarrow$ a one-step def. retraction onto hypercube with dim $n-1$.

+ repeat this process.

□

- Extend the corollary to many more graphs:



Theorem G a graph, $K_1, K_2 \neq \emptyset$, induced subgraphs of G st:

- $V(G) = V(K_1) \amalg V(K_2)$
- $V(K_1) \cap V(K_2) = \emptyset$

Suppose $\exists\ a \in V(K_1)$, $b \in V(K_2)$ st:

- $\{a, b\} \in E(G)$
- $v \in K_1 \Rightarrow \{v, b\} \in E(G)$ "every vertex in K_1 is adjacent to b."
- $v \in K_2 \Rightarrow \{v, a\} \in E(G)$ "every vertex in K_2 is adjacent to a."

Then... $H_n(G) \cong (0)\ \forall n > 0$.

Proof Let $H = \underset{a}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{b}{\bullet} \subseteq G$ be a subgraph.

Define $r: V(G) \to V(H)$ by:

$$r(x) := \begin{cases} a & \text{if } x \in K_2 \setminus \{b\} \\ b & \text{if } x \in K_1 \setminus \{a\} \\ x & \text{if } x \in H \end{cases}$$

$\rightsquigarrow$ a one-step deformation retraction of G onto H.

$\Rightarrow H_n(G) \cong H_n(H)\ \ \forall n > 0$

$\Rightarrow H_n(G) = (0)\ \ \forall n > 0$. $\square$

- There are more operations under which homology is well-behaved. Let's look at those now.

Cor Let K_1, K_2 be graphs with $V(K_1) \cap V(K_2) = \emptyset$.
Let $G := K_1 * K_2$ (the "<u>JOIN GRAPH</u>") where

$$V(G) := V(K_1) \amalg V(K_2)$$
$$E(G) := E(K_1) \cup E(K_2) \cup \{ \{p, q\} : p \in V(K_1),\ q \in V(K_2)\}.$$

Then, $H_n(K_1 * K_2) \cong (0)\ \forall\ n > 0$.

def: The <u>DISJOINT SUM</u> of graphs K_1 and K_2 is the graph $K_1 \oplus K_2$ with

$$V(K_1 \oplus K_2) = V(K_1) \cup V(K_2)$$
$$E(K_1 \oplus K_2) = E(K_1) \cup E(K_2)$$

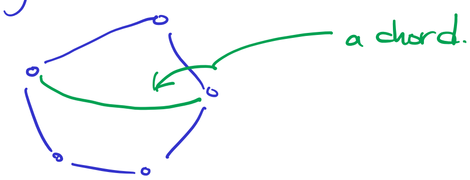
Theorem For any graphs K_1, K_2:

$$\mathcal{H}_n(K_1 \oplus K_2) \cong \mathcal{H}_n(K_1) \oplus \mathcal{H}_n(K_2) \qquad \forall\, n \geq 0.$$

"straightforward"

- There is one final family of graphs that have trivial homology, and they're the collection of CHORDAL GRAPHS.

def: A graph G is CHORDAL if every cycle of length >3 contains a "chord" – edge btw nonadjacent cycle vertices



Thm If G is chordal, then $\mathcal{H}^n(G) \cong (0)$ $\forall\, n>0$.

Proof By homotopy invariance.
OR directly by induction. $\square$

- Homology also behaves nicely wrt the BOX PRODUCT . . .

Thm G, K graphs with H an induced subgraph of K.
Let $r: V(K) \to V(H)$ be a retraction of K onto H. Then,

$$\mathcal{H}_n(G \otimes K) \cong \mathcal{H}_n(G \otimes H) \qquad \forall\, n \geq 0$$

proof Show that $(g,k) \longmapsto (g, r(k))$ is a retraction $G \otimes K \xrightarrow{r} G \otimes H$

(ie) a graph map s.t.

$$r(g,h) = (g, r(h))$$
$$= (g,h)$$

$\Rightarrow$ a retraction $\square$

④ Vanishing homology

- The higher homology groups are trivial for many graphs.
- In particular, one such collection is the collection of all simple graphs without 3- or 4-cycles.

Theorem If G is a graph without 3- or 4-cycles, then $\mathcal{H}_n(G) = (0)$ $\forall\, n \geq 2$.

(Barcelo et al., 2020)

- I'm going to describe the general ideas that go into this result.

(PROOF FOR C_5) $\rightsquigarrow$ (EXTEND TO ALL G WITHOUT 3- OR 4-CYCLES)

- uses a "subdivision map"

- uses covering space theory — that's why we have the hypothesis that G shouldn't have 3- or 4-cycles. This is exactly what was needed for the "lifting lemma". (lift to "universal cover.")
- Maybe the stuff Ublit has done can also extend this result.
- present the full proof at some point?