# Lecture 4: Overview of path homology of digraphs

Mohabat

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# 1 Curvature

## 1.1 Examples of computation of curvature

- in $\mathcal{R}_p$, fix the natural inner product
- using the orthonormal basis $\{e_i\}$ in $\Omega_0 := \langle e_i \rangle_{i \in V}$, we obtain

$$[x, \Omega_0] = \sum_i [x, e_i] = 1$$

- using the orthonormal basis $\{e_{ij}\}$ with $i \to j$ in $\Omega_1 := \langle e_{ij} \rangle_{i \to j}$, we obtain

$$[x, \Omega_1] = \sum_{i \to j} [x, e_{ij}] = \deg(x)$$

- therefore,

$$K_x^{(1)} = 1 - \frac{\deg(x)}{2}$$

and for any $N \geq 1$, we have:

$$K_x^{(N)} = 1 - \frac{\deg(x)}{2} + \sum_{p=2}^{N} \frac{(-1)^p}{p+1}[x, \Omega_p]$$

- By proposition 1.2, in the absence of double arrows, the space $\Omega_2$ always has a basis of triangles and squares
  - but this basis is not necessarily *orthogonal*
- for a triangle $e_{abc} \in \Omega_2$, we have

$$[x, e_{abc}] = \begin{cases} 1 & x \in \{a, b, c\} \\ 0 & \text{else} \end{cases}$$

- for a square $e_{abc} - e_{ab'c} \in \Omega_2$,

$$[x, e_{abc} - e_{ab'c}] = \begin{cases} 2 & x \in \{a, c\} \\ 1 & x \in \{b, b'\} \\ 0 & \text{else} \end{cases}$$

- in particular, if $G$ has no squares, then $\Omega_2$ has a basis $\{\omega_k\}$ that consists of all triangles in $G$
  - this basis is orthonormal and

$$[x, \Omega_2] = \sum_k [x, \omega_k] = \deg_{\triangle}(x) := \text{\# triangles containing } x$$

    * follows that:

$$K_x^{(2)} = 1 - \frac{\deg(x)}{2} + \frac{\deg_{\triangle}(x)}{3}$$

    and this matches (4.2)

### 1.1.1 Examples

1. $G$: a line digraph
2. $G$: a cyclic digraph
3. $G$: a dodecahedron (with any orientation of edges)
4. $G$: a triangle
5. $G$: a square
6. $G$: an $n$-simplex
7. $G$: a bipyramid
8. $G$: a 3-cube
9. $G$: an octahedron

## 1.2 Digraphs of constant curvature

- a graph is **regular** if $\deg(x)$ is constant

*Definition* 1.1. A digraph $G$ is **strongly regular** if $x \mapsto [x, \Omega_p]$ is constant for any $p$.

In particular, $G$ is **regular** because $\deg(x) = [x, \Omega_1]$

- in the strongly regular case, the function $x \mapsto K_x$ is also constant and we set

$$K(G) := K_x = \frac{\chi(G)}{|V|}$$

- $G$: a digraph
- $m \in \mathbb{N}$
- construct a new digraph by adding $m$ new vertices $\{y_1, \dots, y_m\}$ and for all $x \in X$, all arrows $x \longrightarrow y_i$ to the graph $G$
  - this digraph is the $m$**-suspension of** $G$, denoted $\mathrm{sus}_m G$

**Theorem 1.2.** *(4.4)*

*Let G be a strongly regular digraph such that for some $k, m \in \mathbb{N}$ and any $p \geq 0$,*

$$\dim \Omega_p(G) = \binom{k}{p+1} m^{p+1} \qquad (binom(k,m))$$

*Then, $sus_m G$ is also strongly regular, and for all $p \geq 0$,*

$$\dim \Omega_p(sus_m G) = \binom{k+1}{p+1} m^{p+1} \qquad (binom(k+1,m))$$

- for $G$ in the above theorem, we have

$$\begin{aligned} \chi(G) &= \sum_{p \geq 0} (-1)^p \dim \Omega_p \\ &= \sum_{p=0}^{k-1} (-1)^p \binom{k}{p+1} m^{p+1} \\ &= -\sum_{j=1}^{k} (-1)^j \binom{k}{j} m^j \\ &= 1 - (1-m)^k \end{aligned}$$

- it follows that:

$$
\begin{aligned}
K(G) &= \frac{\chi(G)}{|V|} \\
&= \frac{\chi(G)}{\dim \Omega_0} \\
&= \frac{1-(1-m)^k}{km}
\end{aligned}
$$

- of course, the same formula is true for $K(\text{sus}_m G)$ with $k$ replaced by $k+1$:

$$
K(\text{sus}_m G) = \frac{1-(1-m)^{k+1}}{(k+1)m}
$$

### 1.2.1 Example

- a triangle

## 1.3 Some problems

Problem 4.5: Compare this notion of curvature with other definitions of the curvature of graphs.

Problem 4.6: Compute the curvature of the icosahedron.

Problem 4.7: Is it true that for the icosahedron, $|\Omega_2| = 25$ for any numbering of the vertices?

Problem 4.8: Let a digraph $G$ be determined by a triangulation of $\mathbb{S}^2$ (section 2.3). Assume that $\deg(x) \leq 6$ for all vertices $x \in G$. Is it true that $K_x(G) \geq 0$?

Problem 4.9: What can be said about the curvature of a triangulation of $\mathbb{T}^2$?

Problem 4.10: What can be said about the curvature of random digraphs?

Problem 4.11: Let $\mathcal{S}$ be a simplicial complex and $G_{\mathcal{S}}$ its Hasse diagram (section 2.2). Is there any relationship between $K_x(G_{\mathcal{S}})$ and properties of $\mathcal{S}$? For example, we have

$$
K_{\text{total}}(G_{\mathcal{S}}) = \chi(G_{\mathcal{S}}) = \chi_{\text{simp}}(\mathcal{S})
$$

Can one give an explicit formula for computing $K_\sigma(G_{\mathcal{S}})$ for any simplex $\sigma \in \mathcal{S}$?