Lecture 2: Overview of path homology of digraphs

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Contents

1 Path homology of digraphs

1.1 Dependence on the field $\mathbb{K}$

- dimensions $|\Omega_0| = |V|$ and $|\Omega_1| = |E|$ do not depend on the choice of a field $\mathbb{K}$
- using a geometric characterization of Ω_2 in Prop 1.2, we see that $|\Omega_2|$ is also independent of $\mathbb{K}$

Conjecture 1.4: $|\Omega_p|$ is independent of K for any p (a priori $|\Omega_p|(G, \mathbb{Q}) \leq |\Omega_p|(G, \mathbb{F}_q)$)

- let's turn to $|H_p|$
- easy to show: $|H_0| = c$ where c is the number of connected components of G, and hence is independent of $\mathbb{K}$

  Conjecture 1.5: $|H_1|$ is independent of $\mathbb{K}$.

Approach to the proof:

- by linear algebra: $|H_1| = |\Omega_1| - |\partial\Omega_1| - |\partial\Omega_2|$
  - $|\Omega_1| = |E|$
  - $|\partial\Omega_1| = |V| - c$
- i.e., the first 2 terms are independent of $\mathbb{K}$
- remains to verify: $|\partial\Omega_2|$ is independent of $\mathbb{K}$

Remark 1.1. Recall: for *manifolds*: $|H_p|$ may depend on $\mathbb{K}$, e.g.,

$$|H_2|(\mathbb{RP}^2, \mathbb{Q}) = 0 < 1 = |H_2|(\mathbb{RP}^2, \mathbb{F}_2)$$

1.1.1 Example

- there is a randomly generated digraph G that is a candidate for $|H_2|(G, \mathbb{Q}) < |H_2|(G, \mathbb{F}_2)$:
- for this digraph, we have:
  - $|V| = 20$
  - $|E| = 69$
  - $\dim \Omega_2 = 71$

  $$|H_1|(G, \mathbb{F}_2) = |H_1|(G, \mathbb{Q}) = 2$$

  and

  $$|H_2|(G, \mathbb{F}_2) = 5$$

  Conjecture 1.6: For this digraph, $|H_2|(G, \mathbb{Q}) = 4 < 5$

- motivation for this conjecture is as follows
- one of 5 generators of $H_2(G, \mathbb{F}_2)$ is:

  $u = (e_{8318} + e_{81518}) + e_{81519} + e_{91018} + e_{91019} + e_{10318} + e_{1483} + (e_{14819} + e_{141019}) + e_{14103} + e_{15918} + e_{15919}$

By changing the signs of the terms appropriately, we obtain the following element of $H_2(G, \mathbb{Q})$:

$\tilde{u} = (e_{8318} - e_{81518}) + e_{81519} - e_{91018} + e_{91019} - e_{10318} - e_{1483} + (e_{14819} - e_{141019}) + e_{14103} - e_{15918} + e_{15919}$

- the same method works for 4 out of 5 generators of $H_2(G, \mathbb{F}_2)$.
- the fifth generator is:

$$e_{073} + e_{083} + e_{326} + e_{327} + e_{3187} + e_{5148} + e_{8153} + e_{81519} + (e_{907} + e_{9187}) + e_{1427} + e_{1473} +$$
$$e_{91018} + (e_{905} + e_{9115}) + (e_{91113} + e_{91913}) + e_{91019} + e_{10318} + e_{111315} + (e_{1326} + e_{1356})$$
$$e_{058} + e_{14103} + (e_{14819} + e_{141019}) + (e_{1536} + e_{1556}) + (e_{15514} + e_{151914}) + (e_{19132} + e_{19142})$$

  - but for this generator, changing the signs *does not work*

Conjecture 1.7: It is always possible to choose bases in $\Omega_p(G, \mathbb{Q})$ and $H_p(G, \mathbb{Q})$ so that each element of the basis has the form

$$\sum w^{i_0 \dots i_p} e_{i_0 \dots i_p}$$

with $w^{i_0 \dots i_p} \in \{\pm 1, 0\}$

Conjecture 1.8: A basis in $\Omega_p(G, \mathbb{F}_3)$ (resp. $H_p(G, \mathbb{F}_3)$) is also a basis in $\Omega_p(G, \mathbb{Q})$ (resp. $H_p(G, \mathbb{Q})$). In particular, the Betti numbers over $\mathbb{F}_3$ and $\mathbb{Q}$ are the same.

2 Connection to simplexes

2.1 Path complex

- notion of *path complex* unifies **digraphs** and **simplicial complexes**

Definition 2.1. A **path complex** on a finite set V is a collection $\mathcal{P}$ of elementary paths on V such that

$$i_0 i_1 \dots i_{p-1} i_p \in \mathcal{P} \Rightarrow i_1 \dots i_p \text{ and } i_0 \dots i_{p-1} \in \mathcal{P}$$

- e.g., each digraph $G = (V, E)$ gives rise to a path complex $\mathcal{P}$ that consists of all allowed elementary paths, i.e., of the paths $i_0 \to i_1 \to \dots \to i_p$
  - in general, all paths in a path complex $\mathcal{P}$ are also called *allowed*
- the above definitions of ∂-invariant paths, spaces Ω_p and H_p go through without any change to general path complexes in place of digraphs because they are based on the notion of allowed paths only
- for comparison, let's recall the definition of an abstract simplicial complex

Definition 2.2. A **simplicial complex** with the set of vertices V is a collection $\mathcal{S} \subseteq 2^V$ of subsets of V such that if $\sigma \in \mathcal{S}$, then any subset of σ is also an element of S.

- enumerate all elements of V so that any subset $\sigma \subseteq V$ can be regarded as a path $i_0 \dots i_p$ with $i_0 < i_1 < \dots < i_p$

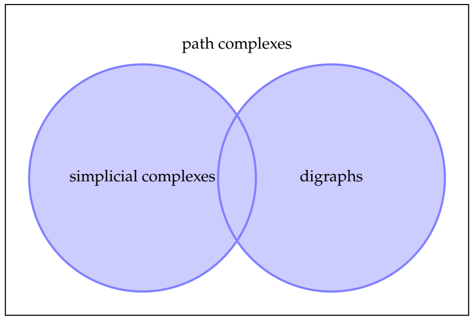


Figure 1: visualizing the connection between digraphs, simplicial complexes, and path complexes

- above definition means that if $i_0 \dots i_p \in \mathcal{S}$, then any **sub-path** $i_{k_0} \dots i_{k_p} \in \mathcal{S}$ as well (where $0 \le k_0 < k_1 < \dots < k_q \le p$)
    - thus, a simplicial complex $\mathcal{S}$ is a path complex, and the theory of path homologies applies to $\mathcal{S}$
- in this case, $\mathcal{A}_p$ consists of linear combinations of all p-dimensional simplexes in $\mathcal{S}$ and $\Omega_p = \mathcal{A}_p$ because $\partial e_{i_0 \dots i_p}$ is always allowed if $e_{i_0 \dots i_p}$ is allowed

  $\Rightarrow$ the *path homology theory of a path complex $\mathcal{S}$ coincides with the simplicial homology theory of $\mathcal{S}$*

2.2 Hasse diagram

- $\mathcal{S}$: a simplicial complex with vertex set V

Definition 2.3. The **Hasse diagram** of $\mathcal{S}$ is the digraph $G_{\mathcal{S}}$ with:

- $V(G_{\mathcal{S}}) = \mathcal{S}$
- $\sigma \longrightarrow \tau$ for $\sigma, \tau \in \mathcal{S}$ if:
    - $\tau \subset \sigma$, and
    - $|\tau| = |\sigma| - 1$
- i.e., $\sigma \longrightarrow \tau$ iff τ is a face of σ of codimension 1.

- if $\mathcal{S}$ is realized geometrically as a collection of simplexes in $\mathbb{R}^n$, then $G_{\mathcal{S}}$ can be realized with the set of vertices $B_{\mathcal{S}}$ consisting of **barycenters** of the simplexes of $\mathcal{S}$ as in Figure 2.2.

Theorem 1. *(2.1)*

$$H_*^{simpl}(\mathcal{S}) \simeq H_*(G_{\mathcal{S}})$$

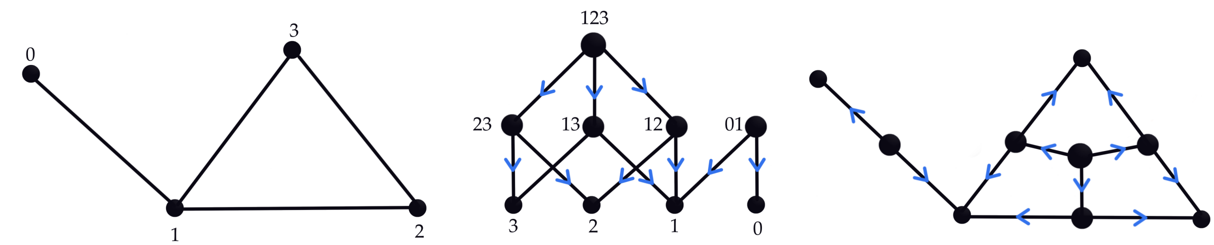


Figure 2: From left to right: a simplicial complex $\mathcal{S}$; abstract digraph $G_{\mathcal{S}}$; and the digraph $G_{\mathcal{S}}$ based on $B_{\mathcal{S}}$

2.3 Triangulation as a closed path

- M: a closed oriented n-dimensional manifold
- T: its *triangulation*
  - i.e., a partition into n-dimensional simplexes
- $V = \{0, 1, \dots\}$: set of all vertices of the simplexes from T
- E: set of all edges of simplexes of T
  - $\rightsquigarrow$ (V, E) is a graph embedded on M
- introduce *directed edges* $i \longrightarrow j$ if $i < j$ and $j \longrightarrow i$ if $i > j$
  - then each simplex from T becomes a **digraph-simplex**
- $\overrightarrow{T}$: the set of all digraph simplexes constructed in this way
  - i.e., $i_0 \dots i_n \in \overrightarrow{T}$ if $i_0 \dots i_n$ is a monotone sequence that determines a simplex from T
    - notice that any such path $i_0 \dots i_p$ is *allowed*
- for any simplex from T with vertices $i_0 \dots i_n$, define:

$$\sigma^{i_0 \dots i_n} := \begin{cases} 1 & \text{if the orientation of the simplex } i_0 \dots i_n \text{ matches the orientation of the manifold } M \\ -1 & \text{else} \end{cases}$$

Then, consider the following allowed n-path on the digraph $G = (V, E)$:

$$\sigma := \sum_{i_0 \dots i_n \in \overrightarrow{T}} \sigma^{i_0 \dots i_n} e_{i_0 \dots i_n} \tag{2.1}$$

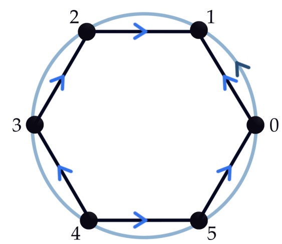
Lemma 2.4. *The path σ is closed (i.e., $\partial\sigma = 0$). In particular, σ is ∂-invariant.*

- the closed paths σ defined by (2.1) is called a **surface path** on M
  - there are several examples of surface paths σ that are also exact (i.e, $\sigma = \partial v$ for some $(n+1)$-path v)

- in this case, v is called a **solid path** on M because v represents a *solid shape* whose boundary is given by a surface path
- if σ is *not exact*, then σ determines a non-trivial homology class from $H_n(G)$ and hence represents a "cavity" in the triangulation T.

2.3.1 Example: $M = \mathbb{S}^1$

- a triangulation of $\mathbb{S}^1$ is a polygon and the corresponding digraph G is **cyclic** as below:



- on each edge (i, j) of a polygon, we choose an arrow $i \to j$ arbitrarily (not necessarily if $i < j$)
- set

$$\sigma^{ij} := \begin{cases} 1 & \text{if the arrow } i \to j \text{ goes counterclockwise} \\ -1 & \text{else} \end{cases}$$

We have:

$$\sigma = \sum_{i \to j} \sigma^{ij} e_{ij}$$

For the digraph in the picture, then we have:

$$\sigma = e_{01} - e_{21} + e_{23} + e_{34} - e_{54} + e_{50}$$

Proposition 2.5.

1. *If a polygon G is not a triangle or square, then*

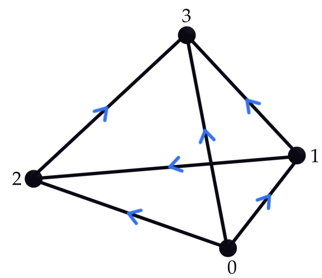
$$\Omega_p = \{0\} \text{ for all } p \geq 2$$
$$H_1 = \langle \sigma \rangle$$
$$H_p = \{0\} \text{ for all } p \geq 2$$

2. *If G is a triangle or a square, then*

$$\Omega_p = \{0\} \text{ for all } p \geq 3$$
$$H_p = \{0\} \text{ for all } p \geq 1$$

2.3.2 Example: $M = \mathbb{S}^n$

- let the triangulation of $\mathbb{S}^n$ be given by an $(n+1)$-simplex
- then, G is an $(n+1)$-simplex digraph as below:



- for this picture, $n = 2$ and

$$\sigma = e_{123} - e_{023} + e_{013} - e_{012} = \partial e_{0123}$$

  so that e_{0123} is a *solid path* representing a tetrahedron

- in general, we also have:

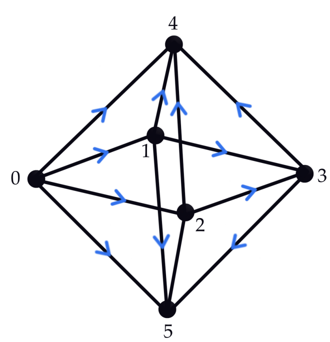
$$\sigma = \partial e_{0 \dots n+1}$$

  so that $e_{0 \dots n+1}$ is a solid path representing an $(n+1)$-simplex

2.3.3 Example: $M = \mathbb{S}^2$, octahedron.

- consider a triangulation of $\mathbb{S}^2$ using an **octahedron** labelled in 2 different ways:

  1. Case A:

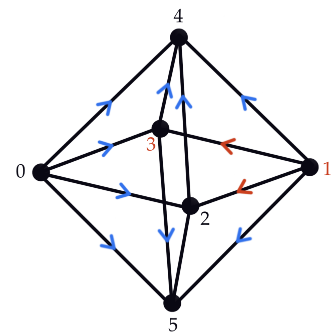


  - $H_2 = \{0\}$

$$\begin{aligned}\sigma &= e_{024} - e_{025} - e_{014} + e_{015} - e_{234} + e_{235} + e_{134} - e_{135} \\ &= \partial \left(e_{0134} - e_{0234} + e_{0135} - e_{0235}\right)\end{aligned}$$

  hence, $v := e_{0134} - e_{0234} + e_{0135} - e_{0235}$ is a solid path and the octahedron represents a **solid shape**

2. Case B:



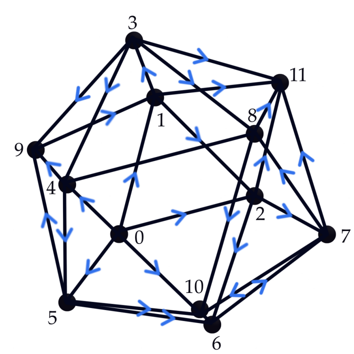
– $H_2 = \langle\, \sigma \,\rangle$

$$\sigma = e_{024} - e_{034} - e_{025} + e_{035} - e_{124} + e_{134} + e_{125} - e_{135}$$

hence the octahedron represents a **cavity**

2.3.4 Example: $M = \mathbb{S}^2$, icosahedron.

- consider an **icosahedron** as a triangulation of $\mathbb{S}^2$ as below:



- we have $H_2 = \langle\, \sigma \,\rangle$ where

$$\begin{aligned}\sigma = &-e_{019} + e_{012} - e_{1211} + e_{026} + e_{059} - e_{056} + e_{5610} - e_{139} + e_{1311} - e_{267}\\ &+ e_{6710} - e_{2711} - e_{349} + e_{348} - e_{4810} + e_{3811} - e_{459} + e_{4510} + e_{7810} - e_{7811}\end{aligned}$$

hence, the icosahedron represents a **cavity**.

Conjecture 2.4: For the icosahedron, $\dim H_2(G) = 1$ ($\Leftrightarrow$ σ is not a boundary) independent of the numbering of the vertices.

Conjecture 2.5: For a general triangulation of $\mathbb{S}^n$, the homology group $H_n(G)$ is either trivial or generated by σ. All other homology groups $H_p(G)$ are trivial.

2.4 Computational challenge

- interesting paper in computational neuroscience: "Cliques of neurons bound into cavities provide a missing link between structure and function"
- they reconstruct a microcircuit from a rat brain as a graph (neurons and connections between them)
- the size of the graph is $|V| \sim 31,000$ and $|E| \sim 8,000,000$. They:
  1. detect cliques in this graph
  2. form a simplicial complex using these cliques
  3. compute the Betti numbers over $\mathbb{F}_2$
- they were able to compute the 5th Betti number β_5 and show that $\beta_5 > 0$

Problem 2.6: Create computational tools capable of computing low-dimensional Betti numbers for path homologies of digraphs of similar size.

- at present, our program can compute:
  - β_1 on a digraph with $|V| \sim 7000$ and $|E| \sim 100,000$
  - β_2 on a digraph with $|V| \sim 4000$ and $|E| \sim 25,000$

- notes will be posted on the speaker's website and updated after each seminar