Lecture 1: Overview of path homology theory of digraphs

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Seminar on path homology of digraphs

Speaker: Alexander Grigor'yan

> Goal of the seminar: To elucidate the current state of research in homology of digraphs and to outline open problems in the area.

- full notes will be available at the end of the seminar series on the speaker's website.

Overview of topics in the seminar:

1. Path homology of digraphs
   - paths in a finite set
   - chain complex and path homology of a digraph
   - examples of ∂-invariant paths
   - examples of spaces Ω_p and H_p
   - example computation of Ω_p and H_p
   - structure of Ω_p
   - dependence on the field $\mathbb{K}$

2. Connection to simplexes
    - path complex
    - Hasse diagrams
    - triangulation as a closed path
    - computational challenge
3. Homological dimension
    - an example of a digraph with infinite homological dimension
    - random digraphs
    - homological dimension and degree
    - homologically trivial and spherical digraphs
    - computational limitations
4. Curvature of digraphs
    - motivation
    - curvature operator
    - examples of computation of curvature
    - digraphs of constant curvature
5. Homology and Cartesian product of digraphs
    - cross product of paths
    - Cartesian product of digraphs
    - Künneth formula
6. Path cohomology
    - exterior derivative

1 Path homology of digraphs

1.1 Paths in a finite set

Definition 1.1. V a finite set and $p \geq 0$.

- An **elementary** p**-path** is any sequence $i_0, i_1, \ldots, i_p$ of $(p+1)$ vertices of V – there *may* be repeated elements.
- $\mathbb{K}$: a field
    - $\Lambda_p := \Lambda_p(V, \mathbb{K})$ is the $\mathbb{K}$-linear space that consists of all formal $\mathbb{K}$-linear combinations of elementary p-paths in V. Elements of Λ_p are called p**-paths**.

- elementary p-path $i_0, i_1, \ldots, i_p \in \Lambda_p$ is denoted $e_{i_0 \ldots i_p}$
  - e.g., we have:

$$\begin{aligned}\Lambda_0 &= \langle\, e_i : i \in V \,\rangle \\ \Lambda_1 &= \langle\, e_{ij} : i, j \in V \,\rangle \\ \Lambda_2 &= \langle\, e_{ijk} : i, j, k \in V \,\rangle\end{aligned}$$

- then any p-path $u \in \Lambda_p$ can be written in a form

$$u = \sum_{i_0, i_1, \ldots, i_p \in V} u^{i_0 i_1 \ldots i_p} e_{i_0 i_1 \ldots i_p}$$

  where $u^{i_0 i_1 \ldots i_p} \in \mathbb{K}$.

Definition 1.2. Let $p \geq 1$. A linear **boundary operator** $\partial : \Lambda_p \to \Lambda_{p-1}$ is

$$\partial e_{i_0 \ldots i_p} := \sum_{q=0}^{p} (-1)^q e_{i_0 \ldots \hat{i_q} \ldots i_p}$$

where $\hat{i_q}$ denotes omitting the index i_q.

- For $p = 0$, $\partial e_i := 0$
  - in particular, $\Lambda_{-1} = \{0\}$

- e.g:
  - $\partial e_{ij} = e_j - e_i$
  - $\partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}$

Lemma 1.3. $\partial^2 = 0$

Proof. For any $p \geq 2$, we have:

$$\begin{aligned}\partial^2 e_{i_0 \ldots i_p} &= \sum_{q=0}^{p} (-1)^q \partial e_{i_0 \ldots \hat{i_q} \ldots i_p} \\ &= \sum_{q=0}^{p} (-1)^q \left(\sum_{r=0}^{q-1} (-1)^r e_{i_0 \ldots \hat{i_r} \ldots \hat{i_q} \ldots i_p} + \sum_{r=q+1}^{p} (-1)^{r-1} e_{i_0 \ldots \hat{i_q} \ldots \hat{i_r} \ldots i_p} \right) \\ &= \sum_{0 \leq r < q \leq p} (-1)^{q+r} e_{i_0 \ldots \hat{i_r} \ldots \hat{i_q} \ldots i_p} - \sum_{0 \leq q < r \leq p} (-1)^{q+r} e_{i_0 \ldots \hat{i_q} \ldots \hat{i_r} \ldots i_p}\end{aligned}$$

By switching q and r in the second sum, we see that the two sums cancel and hence

$$\partial^2 e_{i_0 \ldots i_p} = 0$$

Thus, $\partial^2 u = 0$ for all $u \in \Lambda_p$. □

By this lemma, we have a **chain complex** $\Lambda_*(V)$:

$$0 \longleftarrow \Lambda_0 \xleftarrow{\partial} \Lambda_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} \Lambda_{p-1} \xleftarrow{\partial} \Lambda_p \xleftarrow{\partial} \cdots$$

Definition 1.4. An elementary p-path $e_{i_0 \dots i_p}$ is **regular** if there are no consecutive repeats, i.e., $i_k \neq i_{k+1}$ for all $k = 0, 1, \dots, p-1$. Otherwise, it is **irregular**.

- $I_p := \langle\, e_{i_0 \dots i_p} : e_{i_0 \dots i_p} \text{ is irregular} \,\rangle \subset \Lambda_p$: subspace spanned by all irregular p-paths

Claim 1.5. *$\partial I_p \subset I_{p-1}$*

Proof. If $e_{i_0 \dots i_p}$ is irregular, then $i_k = i_{k+1}$ for some k. We have the following:

$$\partial e_{i_0 \dots i_p} = e_{i_1 \dots i_p} - e_{i_0 i_2 \dots i_p} + \cdots + (-1)^k e_{i_0 \dots i_{k-1} i_{k+1} i_{k+2} \dots i_p} + (-1)^{k+1} e_{i_0 \dots i_{k-1} i_k i_{k+2} \dots i_p} + \cdots + (-1)^p e_{i_0 \dots i_{p-1}}$$

By the fact that $i_k = i_{k+1}$, the two middle terms cancel. As all other terms are irregular, then

$$\partial e_{i_0 \dots i_p} \in I_{p-1}$$

as required. □

- the claim tells us that ∂ is well-defined on the quotient spaces $\mathcal{R}_p := \Lambda_p / I_p$ and so we obtain the **regular chain complex** $\mathcal{R}_*(V)$:

$$0 \longleftarrow \mathcal{R}_0 \xleftarrow{\partial} \mathcal{R}_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} \mathcal{R}_{p-1} \xleftarrow{\partial} \mathcal{R}_p \xleftarrow{\partial} \cdots$$

- by setting all irregular p-paths to be 0, we can identify $\mathcal{R}_p$ with the subspace of Λ_p spanned by all regular paths
  - e.g., if $i \neq j$, then $e_{iji} \in \mathcal{R}_2$ and as $e_{ii} = 0$, we have:

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij}$$

1.2 Chain complex and path homology of a digraph

Definition 1.6. A **digraph (*directed graph*)** is a pair $G := (V, E)$ of a set of **vertices** V and a set of **directed edges/arrows** $E \subset \{V \times V \setminus \text{diag}\}$ (i.e., we exclude self-loops).

- $(i, j) \in E$ is denoted $i \to j$

Definition 1.7. G: a digraph

An elementary p-path $i_0 \dots i_p$ on V is **allowed** if $i_k \to i_{k+1}$ for all $k = 0, \dots, p-1$ and **non-allowed** otherwise.

$$\mathcal{A}_p := \mathcal{A}_p(G) = \langle\, e_{i_0 \dots i_p} : i_0 \dots i_p \text{ is allowed} \,\rangle$$

is the $\mathbb{K}$-linear space spanned by all allowed elementary p-paths. Its elements are called **allowed p-paths**.

- every allowed path is regular so we have

$$\mathcal{A}_p \subset \mathcal{R}_p \subset \Lambda_p$$

What do we want to do?

- build a chain complex based on the subspaces $\mathcal{A}_p \subset \mathcal{R}_p$
- the issue is that in general, the subspaces $\mathcal{A}_p$ are *not invariant under* ∂, e.g.:

$$a \longrightarrow b \longrightarrow c$$

  - in the above digraph, we have that $e_{abc} \in \mathcal{A}_2$
  - on the other hand, since e_{ac} is not allowed, we have $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \notin \mathcal{A}_1$

- we need a way to get around this invariance issue, so we define the following subspace of $\mathcal{A}_p$:

$$\Omega_p = \Omega_p(G) := \{u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1}\} \subset \mathcal{A}_p$$

Claim 1.8. $\partial \Omega_p \subset \Omega_{p-1}$

Proof. As $u \in \Omega_p$ implies $\partial u \in \mathcal{A}_{p-1}$ and $\partial(\partial u) = 0 \in \mathcal{A}_{p-2}$, thus $\partial u \in \Omega_{p-1}$ as required. □

Definition 1.9. Elements of Ω_p are **∂-invariant p-paths**

- obtain a chain complex $\Omega_* = \Omega_*(G)$:

$$0 \longleftarrow \Omega_0 \xleftarrow{\partial} \Omega_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} \Omega_{p-1} \xleftarrow{\partial} \Omega_p \xleftarrow{\partial} \cdots$$

- by construction: $\Omega_0 = \mathcal{A}_0 = \langle e_i \rangle$ and $\Omega_1 = \mathcal{A}_1 = \langle e_{ij} : i \to j \rangle$
  - in general: $\Omega_p \subset \mathcal{A}_p$

Definition 1.10. The **path homologies of** G are the homologies of the chain complex $\Omega_*(G)$, i.e.,

$$H_p(G) := \ker \partial|_{\Omega_p} / \operatorname{Im} \partial|_{\Omega_{p+1}}$$

The **Betti numbers** of G are the dimensions of the homologies:

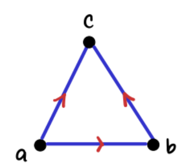
$$\beta_p(G) := \dim H_p(G)$$

- easy to show: $\beta_0(G) = \#\{\text{connected components of } G\}$

1.3 Examples of ∂-invariant paths

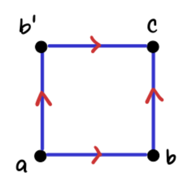
Open problem: Producing interesting ∂-invariant paths.

1. A **triangle** is a sequence of 3 vertices a, b, c such that $a \to b \to c$ and $a \to c$



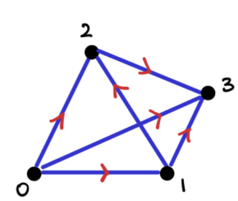
- since $e_{abc} \in \mathcal{A}_2$ and
$$\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1$$
then the triangle determines a 2-path $e_{abc} \in \Omega_2$

2. A **square** is a sequence of 4 vertices a, b, b', c such that $a \to b, b \to c, a \to b'$, and $b' \to c$



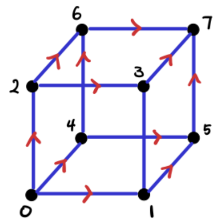
- since $u = e_{abc} - a_{ab'c} \in \mathcal{A}_2$ and
$$\begin{aligned} \partial u &= (e_{bc} - e_{ac} + e_{ab}) - (e_{b'c} - e_{ac} + e_{ab'}) \\ &= e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1 \end{aligned}$$
then the square determines a 2-path $u \in \Omega_2$.

3. A p-**simplex/**p-**clique** is a sequence of $(p+1)$ vertices $0, 1, \ldots, p$ such that $i \to j$ for all $i < j$



- determines a p-path $e_{01\ldots p} \in \Omega_p$

4. A **3-cube** is a sequence of 8 vertices $0, 1, \ldots, 7$ connected by arrows as here:

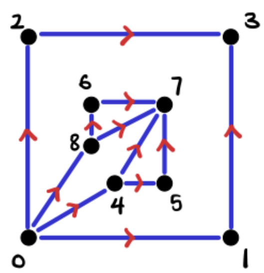


- let $u = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \in \mathcal{A}_3$
- since we have:

  $\partial u = (e_{013} - e_{023}) + (e_{157} - e_{137}) + (e_{237} - e_{267}) - (e_{046} - e_{026}) - (e_{457} - e_{467}) - (e_{015} - e_{045}) \in \mathcal{A}_2$

  then, $u \in \Omega_3$ and the 3-cube determines a ∂-invariant 3-path.

5. An **exotic cube** consists of 9 vertices connected by arrows as follows:



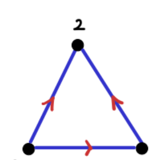
- determines a ∂-invariant 3-path v where:

$$v = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0867} - e_{0267} \in \Omega_3$$

1.4 Examples of spaces Ω_p and H_p

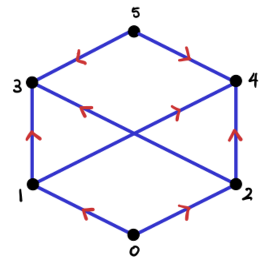
Notation: For a vector space A over $\mathbb{K}$, we write $|A| := \dim_{\mathbb{K}} A$

1. Triangle as a digraph



- $\Omega_1 = \langle e_{01}, e_{02}, e_{12} \rangle$
- $\Omega_2 = \langle e_{012} \rangle = \mathcal{A}_2$
- $\ker \partial|_{\Omega_1} = \langle e_{01} - e_{02} + e_{12} \rangle$ but $e_{01} - e_{02} + e_{12} = \partial e_{012} \in \operatorname{Im} \partial_{\Omega_2}$ so that $H_1 = \{0\}$
- $H_p = \{0\}$ for all $p \geq 2$

2. Hexagon with diagonals

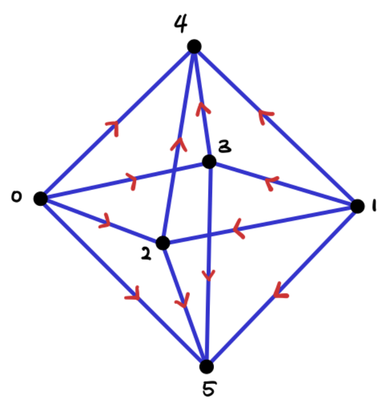


- $|\Omega_0| = 6$
- $|\Omega_1| = 8$
- Ω_2 is spanned by 2 squares:

$$\Omega_2 = \langle\, e_{013} - e_{023}, e_{014} - e_{024} \,\rangle$$

- $\Omega_p = \{0\}$ for all $p \geq 3$
- $H_1 = \langle\, e_{13} - e_{53} + e_{54} - e_{14} \,\rangle$
- $|H_1| = 1$
- $H_p = \{0\}$ for all $p \geq 2$

3. Octahedron



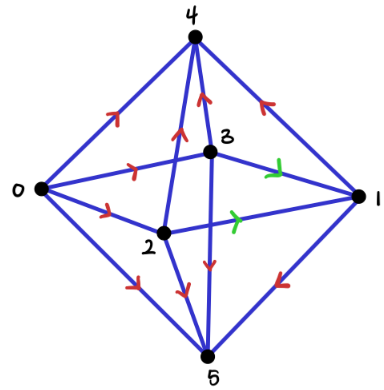
- $|\Omega_0| = 6$
- $|\Omega_1| = 12$
- Ω_2 is spanned by 8 triangles:

$$\Omega_2 = \langle\, e_{024}, e_{034}, e_{025}, e_{035}, e_{124}, e_{134}, e_{125}, e_{135} \,\rangle$$

- $|\Omega_2| = 8$
- $\Omega_p = \{0\}$ for all $p \geq 3$
- $H_2 = \langle\, e_{024} - e_{034} - e_{025} + e_{035} - e_{124} + e_{134} + e_{125} - e_{135} \,\rangle$
- $|H_2| = 1$

- $|H_p| = 0$ for $p = 1$ and $p \geq 3$

4. Octahedron with different orientation



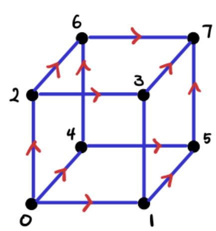
- $\Omega_2 = \langle\, e_{024}, e_{025}, e_{014}, e_{015}, e_{234}, e_{235}, e_{134}, e_{135}, e_{013} - e_{023} \,\rangle$
- $\Omega_3 = \langle\, e_{0234} - e_{0134}, e_{0235} - e_{0135} \,\rangle$
- $|\Omega_2| = 9$
- $|\Omega_3| = 2$
- $\Omega_p = \{0\}$ for all $p \geq 4$
- $\ker \partial_{\Omega_2} = \langle\, u, v \,\rangle$ where:

$$u = e_{024} + e_{234} - e_{014} - e_{134} + (e_{013} - e_{023})$$
$$v = e_{025} + e_{235} - e_{015} - e_{135} + (e_{013} - e_{023})$$

- $H_2 = \{0\}$ because:

$$u = \partial\,(e_{0234} - e_{0134})$$
$$v = \partial\,(e_{0235} - e_{0135})$$

5. 3-cube



- $|\Omega_0| = 8$
- $|\Omega_1| = 12$
- Ω_2 is spanned by 6 squares:

$$\Omega_2 = \langle\, e_{013} - e_{023}, e_{015} - e_{045}, e_{026} - e_{046}, e_{137} - e_{157}, e_{237} - e_{267}, e_{457} - e_{467} \,\rangle$$

  - hence $|\Omega_2| = 6$

- Ω_3 is spanned by one 3-cube:

$$\Omega_3 = \langle\, e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \,\rangle$$

  - hence $|\Omega_3| = 1$
- $|\Omega_p| = 0$ for all $p \geq 4$
- $|H_p| = 0$ for all $p \geq 1$

Remark 1.11. Computations of spaces $\Omega_p(G)$ and $H_p(G)$ amount to computing ranks and null spaces of large matrices. For numerical computation of $H_p(G, \mathbb{F}_2)$, currently use a C++ program written by Chao Chen in 2012.

1.5 Structure of Ω_p

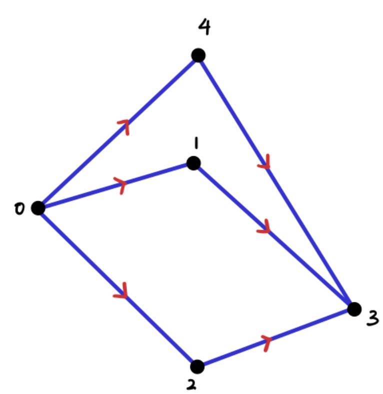
- Recall that $\Omega_0 = \langle\, e_i \,\rangle$ consists of all vertices and $\Omega_1 = \{e_{ij} : i \to j\}$ consists of all arrows
- What about higher Ω_p's?

Proposition 1.12.

1. *The space Ω_2 is spanned by all triangles e_{abc}, squares $e_{abc} - e_{ab'c}$, and double arrows e_{aba}*
2. *$|\Omega_2| = |\mathcal{A}_2| - s$ where s is the number of **semi-arrows** – i.e., pairs of vertices x, y such that $x \not\to y$ but $x \to z \to y$ for some intermediary vertex z.*

- the triangles and double arrows are *always linearly independent*
- the squares can be *dependent*

Example:



For this digraph, we have 3 squares:

$$e_{013} - e_{023}$$
$$e_{043} - e_{013}$$
$$e_{023} - e_{043}$$

but their sum is 0 (so they are *dependent*)

- in this case, $|\Omega_2| = 2 = |\mathcal{A}_2| - s = 3 - 1$

Definition 1.13. X, Y : digraphs

A map of vertex sets $f : X \to Y$ is a **morphism of digraphs** if for any $a \to b \in X$ we have either:

1. $f(a) \to f(b) \in Y$; or
2. $f(a) = f(b)$ in Y

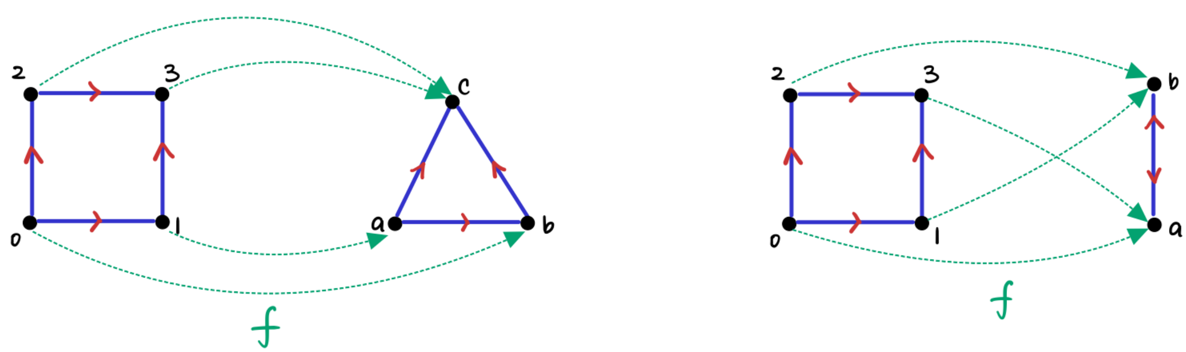
i.e., the image of an arrow is either an arrow or a vertex (same defn we're used to)

The **image** of a path $e_{i_0 \ldots i_p}$ is

$$f\left(e_{i_0 \ldots i_p}\right) := e_{f(i_0) \ldots f(i_p)}$$

so that the image of an allowed path is either allowed or zero (that is also allowed.

- it is clear that $f \circ \partial = \partial \circ f$ so that *morphism images of ∂-invariant paths are also ∂-invariant*
- Some possible morphism images of a square $e_{013} - e_{023}$:



  - a triangle e_{abc}
  - a double arrow e_{aba}

Thus, we can rephrase the proposition as follows:

Proposition 1.14. *(rephrasing of proposition (1))*

*Ω_2 is spanned by squares and their morphism images. i.e., squares are **basic shapes** of Ω_2.*

Open Problem: Describe all basic shapes in Ω_3 and in general, in Ω_p for $p > 3$.