Generating polynomials of exponential random graphs

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Overview

1. Random graphs
2. Encoding dependence in random graphs
3. Strongly Rayleigh Markov random graphs
4. Lorentzian polynomials and distributions

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Random graphs

What are random graphs?

a set of graphs $+$ some (measurable) uncertainty

a set of graphs $+$ a probability distribution

Probability on a finite set

- Ω : a finite set
- $X : \Omega \to \mathbb{R}$: discrete random variable
- $P : \Omega \to [0, 1]$: a probability mass function on Ω
  - $\sum_{\omega \in \Omega} P(X = \omega) = 1$

Random subgraphs

- E : a finite set
- $\mathcal{P}(E)$: power set of E
- A random subset $S \subseteq E$ is a random element of $\mathcal{P}(E)$

Generating polynomials

- Well-developed ([BBL09], [ALGV19]) dictionary

  $$\{\text{multiaffine polynomials}\} \longleftrightarrow \{\text{probability distributions}\}$$

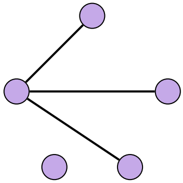
- def. For X a random subset, its generating polynomial is

  $$g_X := \sum_{S \subseteq E} P(X = S)\mathbf{x}^S$$

  where $\mathbf{x}^S = \prod_{i \in S} x_i$ and $\mathbf{x} = (x_e)_{e \in E}$

A foundational random graph model: Erdős-Rényi

- $G = (V, E)$: finite graph
- Erdős-Rényi graphs $G(p)$ for $0 < p < 1$
  1. Start with vertices V



  2. Draw an edge between each pair of vertices with independent probability p
- Limitations: independence

Erdős-Rényi graphs

- For $S \subseteq E$, $P(X = S) = p^{|S|}(1-p)^{|E|-|S|}$

Proposition

If X is Erdős-Rényi, then $g_X = \prod_{e \in E}(px_e + (1-p))$.

Enhancing the dictionary

$$\{\text{multiaffine polynomials}\} \longleftrightarrow \{\text{probability distributions}\}$$

Operations

- Multiplication $\longleftrightarrow$ disjoint union
  - $P(X \sqcup Y = S \sqcup T) := P(X = S)P(Y = T)$
  - $g_X \cdot g_Y = g_{X \sqcup Y}$
- Partial differentiation $\longleftrightarrow$ conditioning
  - $\partial_i g_X = \sum_{S \ni i} P(X = S)\mathbf{x}^{S \setminus \{i\}}$
  - i.e., $X \longmapsto (X \mid i \in S)$
- Specialization $\longleftrightarrow$ conditioning
  - $g_X|_{x_i=0} = \sum_{S \not\ni i} P(X = S)\mathbf{x}^S$
  - i.e., $X \longmapsto (X \mid i \notin S)$

Enhancing the dictionary

Properties

- Positive coefficients $\longleftrightarrow$ positive distribution
- Product of linear factors $\longleftrightarrow$ Erdős-Rényi
- Stable $\longleftrightarrow$ strongly Rayleigh/negative dependence
- Lorentzian $\longleftrightarrow$ a weaker form of negative dependence

Irreducible generating polynomials

- V : finite set
- $E := \binom{V}{k}$: all subsets of V of size k
- $\mathrm{Sym}(V) \subseteq \mathrm{Sym}(E)$

Theorem 1

If $\mathrm{supp}(X) = E$ and g_X is $\mathrm{Sym}(V)$-symmetric, then either:

1. $g_X = \prod_{e \in E}(px_e + (1-p))$ for some $p \in (0,1)$; or
2. g_X is irreducible over $\mathbb{R}[x_e : e \in E]$.

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Dependence

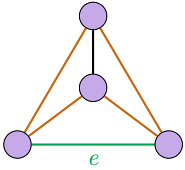
How do we define a joint distribution where the random variables are not independent?

- Markov property:

$$P(X_e = x_e \mid X_{\overline{e}} = x_{\overline{e}}) = P(X_e = x_e \mid X_{N_e} = x_{N_e})$$

- Markov neighbourhood: Adjacent edges are dependent:

$$N_e = \{f \in E : e \sim f\}$$



- Neighbourhood clique $C \subseteq E$: For all $e, f \in C$, $e \sim f$

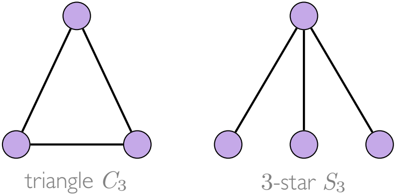
Markov (exponential) random graphs

- $G = (V, E)$: finite (undirected) graph with no self-loops
- $S \subseteq E \rightsquigarrow G_S = (V, S)$ is the spanning subgraph on S
- def. The Markov random graph model on $\mathcal{P}(E)$ is:

$$P(X = G_S) \propto \exp\left(\frac{1}{T}\left(\beta t(C_3, G_S) + \sum_{k\geq 1} \beta_k t(S_k, G_S)\right)\right)$$

▷ $\beta, \beta_k \in \mathbb{R}$, $T > 0$: parameters

▷ $t(H, G_S)$: the homomorphism density of H in G_S



Homomorphism densities

$$P(X = G_S) \propto \exp\left(\frac{1}{T}\left(\beta t(C_3, G_S) + \sum_{k\geq 1}\beta_k t(S_k, G_S)\right)\right)$$

- H, G : connected simple graphs
- The homomorphism density of H in G is
$$t(H, G) := \frac{|\mathrm{Hom}(H, G)|}{|V(G)|^{|V(H)|}}$$
- $|\mathrm{Hom}(S_1, G)| = 2|E(G)|$
- $|\mathrm{Hom}(C_3, G)| = 6 \cdot \#(\text{triangles in } G)$
- Model can be defined for any real-valued function of the degree sequence of G_S (e.g., subgraph counts)

The Markov-Gibbs correspondence

$$\{\text{positive Markov random fields}\} \longleftrightarrow \{\text{finite Gibbs distributions}\}$$

Hammersley-Clifford theorem (1971)

A collection of positive random variables satisfy a Markov property if and only if it is a (finite) Gibbs distribution:

$$P(X = S) \propto \exp\left(-\mathcal{E}(S)\right)$$

where $\mathcal{E}$ is an energy function that encodes the neighbourhood dependencies

- Every finite Gibbs distribution is positive
- Positivity relevant to hypotheses of irreducibility result

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Stable polynomials

- def. A nonzero polynomial $g \in \mathbb{R}[\mathbf{x}]$ is (real) stable if it does not have any roots in the open upper half of the complex plane $\mathcal{H}^E = \{\mathbf{x} \in \mathbb{C}^E : \text{Im}(x_e) > 0 \text{ for all } e\}$.
- def. A probability distribution is strongly Rayleigh if g_X is stable.

Proposition [Brä07]

A multiaffine polynomial $g \in \mathbb{R}[\mathbf{x}]$ is stable if and only if for all $\mathbf{x} \in \mathbb{R}^E$ and $i, j \in E$ such that $i \neq j$,

$$\frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{x}) g(\mathbf{x}) \leq \frac{\partial g}{\partial x_i}(\mathbf{x}) \frac{\partial g}{\partial x_j}(\mathbf{x})$$

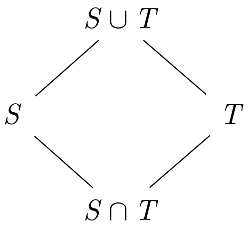
Negative dependence

- Modelling repelling particles
- Pairwise negative correlation: For $i \neq j$,

$$P(i, j \in S) \leq P(i \in S)P(j \in S)$$

- Negative lattice condition: For $S, T \subseteq E$,

$$P(S \cup T)P(S \cap T) \leq P(S)P(T)$$



Negative dependence and stability

Proposition [BBL09]

Strongly Rayleigh probability measures satisfy the strongest form of negative dependence.

- Erdős-Rényi graphs: $g_X = \prod_{e \in E}(px_e + (1-p))$
- Negatively dependent probability measures have applications to determinantal point processes and machine learning ([AGV21, KT12])
- Identifying negatively dependent measures ([Pem00])

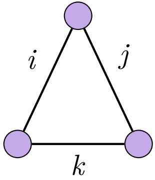
Strongly Rayleigh Markov random graphs: necessary conditions

Theorem 2

If $G = (V, E)$ has at least one triangle and P is a strongly Rayleigh Markov random graph on G, then the triangle and 2-star parameters β and β_2 are such that $\beta \leq -\beta_2$.

Proof:

- Idea: use the negative lattice condition with $S = \{i, j\}$ and $T = \{k\}$ where $S \cup T = \{i, j, k\}$ is a triangle in G



Strongly Rayleigh Markov random graphs: necessary conditions

$$P(S \cup T)P(S \cap T) \leq P(S)P(T)$$

$$P(\triangle)P(\varnothing) \leq P(S_2)P(S_1)$$

- For ease of notation: $X_i := \exp\left(\frac{\beta_i}{Tn^{i+1}}\right)$ and $Y := \exp\left(\frac{\beta}{Tn^3}\right)$
- e.g., $P(\triangle) \propto X_1^6 X_2^{12} Y^6$

$$X_1^6 X_2^{12} Y^6 \leq X_1^4 X_2^6 X_1^2$$

$$Y^6 \leq X_2^{-6}$$

Therefore, $\exp\left(\frac{6\beta}{Tn^3}\right) \leq \exp\left(\frac{-6\beta_2}{Tn^3}\right)$ so that $\beta \leq -\beta_2$.

□

Strongly Rayleigh Markov random graphs: characterizations

Theorem 3

The edge-triangle Markov random graph model on $G = K_3$ is strongly Rayleigh if and only if the triangle parameter $\beta = 0$.

Theorem 4

The edge parameter β_1 in a Markov exponential random graph model on a finite graph G does not affect whether or not the model is strongly Rayleigh.

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Lorentzian polynomials

- def. A subset $J \subseteq \mathbb{N}^E$ is M-convex when it satisfies the symmetric basis exchange property:

  For any $\alpha, \beta \in J$ and an index i such that $\alpha_i > \beta_i$, there exists an index j such that $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$ and $\beta - e_j + e_i \in J$

Examples

- $G = (V, E)$: a finite connected graph
  - $J = \{\text{spanning trees of } G\} \subseteq \{0,1\}^E$
- $M =$ matroid on a finite ground set E
  - $J = \{\text{bases of } M\}$
- $J = \Delta_E^d \subseteq \mathbb{N}^E$
  - d^{th} discrete simplex
  - Vectors with coordinate sum d

Lorentzian polynomials

- def. The support of a polynomial $g \in \mathbb{R}[\mathbf{x}]$ is

$$\mathrm{supp}(g) := \{S \in \mathbb{N}^E : c_S \neq 0\} \subseteq \mathbb{N}^E$$

  where $g(\mathbf{x}) = \sum_{S \in \mathbb{N}^E} c_S \mathbf{x}^S$

- def. A multiaffine polynomial $g \in \mathbb{R}[\mathbf{x}]$ is called positive if $\mathrm{supp}(g) = \{0,1\}^E$

Lorentzian polynomials

- Notation
  - H_E^d : homogeneous polynomials of degree d in variables $(x_e)_{e \in E}$
  - $M_E^d \subseteq H_E^d$: polynomials in H_E^d whose supports are M-convex
  - $L_E^2 \subseteq H_E^2$: quadratic forms with non-negative coefficients that have at most one positive eigenvalue
- def. A homogeneous polynomial $h \in H_E^d$ is Lorentzian if its support is M-convex and $\partial_i h \in L_E^{d-1}$ for all $i \in E$. i.e., for $d > 2$:

$$L_E^d := \{h \in M_E^d : \partial_i h \in L_E^{d-1} \text{ for all } i \in E\}$$

Lorentzian distributions

- def. The homogenization of g_X is

$$h_X(z, \mathbf{x}) := \sum_{S \subseteq E} P(X = S) z^{|E|-|S|} \mathbf{x}^S$$

- def. A probability distribution is Lorentzian if h_X is Lorentzian.

Proposition [BH20]

If P is strongly Rayleigh, then P is Lorentzian.

Lorentzian negative dependence

Proposition [BH20]

Lorentzian probability measures are 2-Rayleigh: for all $\mathbf{x} \in \mathbb{R}^E_{\geq 0}$ and $i, j \in E$ such that $i \neq j$,

$$\frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{x}) g(\mathbf{x}) \leq 2 \left(\frac{\partial g}{\partial x_i}(\mathbf{x}) \frac{\partial g}{\partial x_j}(\mathbf{x}) \right)$$

- Lorentzian is weaker than stability (but easier to test)

Lorentzian Markov random graphs

Proposition

If P is positive, then the support of h_X is M-convex.

- e.g., finite Gibbs distributions on a power set have M-convex support

Lorentzian Markov random graphs: characterizations

Theorem 5

The edge-triangle Markov random graph model on $G = K_3$ is Lorentzian if and only if the triangle parameter $\beta \leq 0$.

Theorem 6

The edge parameter β_1 in a Markov exponential random graph model on a finite graph G does not affect whether or not the model is Lorentzian.

Lorentzian Markov random graphs: edges

Proof:

- Let P be the Markov random graph model on $G = (V, E)$

$$\begin{aligned} h_P(z, \mathbf{x}) &= \sum_{S \subseteq E} P(S) \mathbf{x}^S z^{|E|-|S|} \\ &= \sum_{S \subseteq E} X_1^{2|S|} \prod_{i \geq 2} X_i^{|\mathrm{Hom}(S_i, G_S)|} Y^{|\mathrm{Hom}(C_3, G_S)|} \mathbf{x}^S z^{|E|-|S|} \end{aligned}$$

- Change of variables $z \mapsto zX_1^2$

$$h_P(X_1^2 z, \mathbf{x}) = \sum_{S \subseteq E} X_1^{2|S|} \prod_{i \geq 2} X_i^{|\mathrm{Hom}(S_i, G_S)|} Y^{|\mathrm{Hom}(C_3, G_S)|} \cdot \mathbf{x}^S (zX_1^2)^{|E|-|S|}$$

Lorentzian Markov random graphs: edges

Then,

$$h_P(X_1^2 z, \mathbf{x}) = X_1^{2|E|} \sum_{S \subseteq E} \prod_{i \geq 2} X_i^{|\mathrm{Hom}(S_i, G_S)|} Y^{|\mathrm{Hom}(C_3, G_S)|} \mathbf{x}^S z^{|E|-|S|}$$

- Positive scaling does not affect the signatures of any quadratic forms

□

A Lorentzian algorithm for Markov random graphs

- Determining whether the Lorentzian property holds is about identifying the signature of symmetric matrices
- Uses the Schur complement of a symmetric block matrix
- Can be generalized and implemented for $G = K_n$

Lorentzian characterizations for $G = K_4$

- Using subgraph counts instead of homomorphism densities

Theorem 7

Suppose P is the Markov random graph model on K_4 and the 2-star parameter $\beta_2 = 0$. Then, P is Lorentzian if and only if the 3-star and triangle parameters β_3, β satisfy

$$T \ln\left(\frac{3-\sqrt{3}}{4}\right) \leq \beta_3 \leq 0 \text{ and}$$

$$-T \ln 2 - \beta_3 \leq \beta \leq 0$$

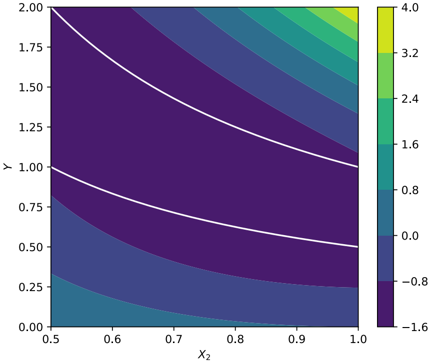
Lorentzian characterizations for $G = K_4$

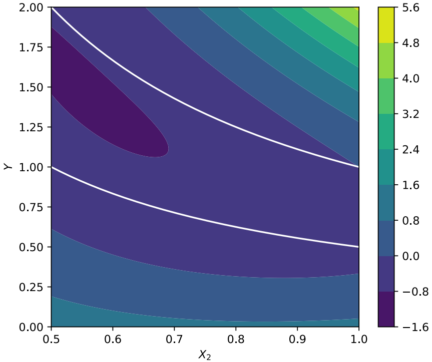
Theorem 8

Suppose P is the Markov random graph model on K_4 and the 3-star parameter $\beta_3 = 0$. Then, P is Lorentzian if and only if the 2-star and triangle parameters β_2, β satisfy

$$-T \ln 2 \leq \beta_2 \leq 0 \text{ and}$$
$$-T \ln 2 - \beta_2 \leq \beta \leq -\beta_2$$

Minors for $\beta_3 = 0$





$g_4(X_2, Y) = 3X_2^2 Y^2 + 2X_2^2 Y + 3X_2^2 - 6X_2 Y - 6X_2 + 3$

$g_5(X_2, Y) = 3X_2^2 Y^2 + 2X_2^2 Y + 3X_2^2 - 6X_2 Y - 5X_2 + 3$

Thank you!

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