Chapter 5: Monads and their algebras

Section 5.1. Monads from adjunctions

Exercise 5.1.iii. The adjunction associated to a reflective subcategory C induces an **idempotent monad** on C. Prove that the following three characterizations of an idempotent monad (T, η, μ) are equivalent.

(i) The multiplication $\mu : T^2 \Rightarrow T$ is a natural isomorphism.

(ii) The natural transformations $\eta T, T\eta : T \Rightarrow T^2$ are equal.

(iii) Each component of $\mu : T^2 \Rightarrow T$ is a monomorphism.

- **(i)⇒(iii)**:
  Clearly if each component of μ is an isomorphism, then each component is also a monomorphism.

- **(iii)⇒(ii)**:
  Suppose each component of $\mu : T^2 \Rightarrow T$ is a monomorphism.
  By the definition of a monad, the following commutes in C^{C}:

$$\begin{array}{ccccc} T & \xRightarrow{\eta T} & T^2 & \xLeftarrow{T\eta} & T \\ & {\scriptstyle 1_T}\searrow & \Downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_T} & \\ & & T & & \end{array}$$

  That is,

$$\mu \circ T\eta = 1_T$$
$$\mu \circ \eta T = 1_T$$

  As each component of μ is a monomorphism, then $T\eta = \eta T$.

- **(ii)⇒(i)**:
  Suppose $\eta T = T\eta$.
  Claim: For every T-algebra[1] A, the T-action $a : TA \to A$ is an isomorphism.
  Proof of claim: By the definition of T-algebra, the following commutes in C (by 5.2.5):

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & {\scriptstyle 1_A}\searrow & \downarrow{\scriptstyle a} \\ & & A \end{array}$$

  That is,

$$a \circ \eta_A = 1_A$$

  Thus, by applying the endofunctor T,

$$T(a) \circ T(\eta_A) = 1_{TA}$$

  Since $\eta_{TA} = T\eta_A$, then

$$T(a) \circ \eta_{TA} = 1_{TA}$$

[1]this is defined in 5.2.4.

Since η is natural, then

$$\eta_A \circ a = 1_{TA}$$

As $a \circ \eta_A = 1_A$ and $\eta_A \circ a = 1_{TA}$, then a is an isomorphism. □

By this claim, since every T-action is an isomorphism, the components $\mu_A : T^2 A \to TA$ are isomorphisms as well (they are a type of T-action). In particular, every component of μ is an isomorphism so μ is a natural isomorphism.

□

Section 5.2. Adjunctions from monads

Exercise 5.2.iii. Dualize Definition 5.2.4 and Lemma 5.2.8 to define the category of coalgebras for a comonad together with its associated forgetful–cofree adjunction.

Let C be a category with a comonad (K, ϵ, δ) where $K : \mathsf{C} \to \mathsf{C}$ is an endofunctor, $\epsilon : K \Rightarrow 1_{\mathsf{C}}$, $\delta : K \Rightarrow K^2$. The dual definition of the category of T-algebras is the category of K**-coalgebras**, C^K, given by the following:

- objects: pairs $(A \in \mathsf{C}, a : A \to KA)$ so that the diagrams

$$\begin{array}{ccc} A & \xrightarrow{a} & KA \\ & \searrow^{1_A} & \uparrow{\scriptstyle \eta_A} \\ & & A \end{array} \qquad \begin{array}{ccc} K^2A & \xrightarrow{\mu_A} & KA \\ {\scriptstyle Ka}\uparrow & & \uparrow{\scriptstyle a} \\ KA & \xleftarrow[a]{} & A \end{array}$$

  commute in C, and
- morphisms: $f : (A, a) \to (B, b)$ is a morphism $f : B \to A \in \mathsf{C}$ so that the square

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ KB & \xrightarrow[Kf]{} & KA \end{array}$$

  commutes in C.

The dual to lemma 5.2.8 for comonads is given as follows:

For any comonad (K, ϵ, δ) acting on a category C, there exists an adjunction

$$\mathsf{C} \underset{F^K}{\overset{U^K}{\rightleftarrows}} \mathsf{C}^K \quad (\top)$$

where $U^K : \mathsf{C}^K \to \mathsf{C}$ is the forgetful functor and $F^K : \mathsf{C} \to \mathsf{C}^K$ takes $A \in \mathsf{C}$ to the free K-coalgebra

$$F^K A := (KA, \delta_A : KA \to K^2 A)$$

and takes a morphism $f : A \to B$ to cofree k-coalgebra morphism

$$F^K f := (KB, \delta_B) \to (KA, \delta_A)$$

□

Exercise 5.2.iv. Verify that the Kleisli category is a category by checking that the composition operation of Definition 5.2.9 is associative.

Recall definition 5.2.9: Given a monad (T, η, μ) on a category C. The Kleisli category C_T is a category with objects the objects of C, that is: an object $x \in \mathsf{C}$ is associated with itself, denoted $x_T \in \mathsf{C}_T$. A morphism $f^\flat : x \rightsquigarrow y$ in C_T is a morphism $f : x \to Ty \in \mathsf{C}$. Composition in C_T, for $f^\flat : x \to y$, $g^\flat : y \to z$ is defined by $g^\flat \circ f^b := (\mu_z \circ Tg \circ f)^\flat$. This is associative because of the following commutative diagram.

$$\begin{array}{ccccccc}
x & \xrightarrow{f} & Ty & \xrightarrow{Tg} & T^2z & \xrightarrow{\mu_z} & Tz \\
{\scriptstyle f}\downarrow & & & & & & \downarrow{\scriptstyle Th} \\
Ty & & & & & & T^2w \\
{\scriptstyle Tg}\downarrow & & & & & & \downarrow{\scriptstyle \mu_w} \\
T^2z & \xrightarrow[T^2h]{} & T^3w & \xrightarrow[T\mu_w]{} & T^2w & \xrightarrow[\mu_w]{} & Tw
\end{array}$$

This reduces to the following commutative diagram:

$$\begin{array}{ccc}
T^2z & \xrightarrow{\mu_z} & Tz \\
{\scriptstyle T^2h}\downarrow & & \downarrow{\scriptstyle Th} \\
T^3w & \xrightarrow[\mu_{Tw}]{T\mu_w} & T^2w
\end{array}$$

This is commutative by naturality of μ at $z \xrightarrow{h} Tw$.

$\square$

Section 5.5. Recognizing categories of algebras

Exercise 5.5.iv. For any group G, the forgetful functor $\mathsf{Set}^{\mathsf{B}G} \to \mathsf{Set}$ admits a left adjoint that sends a set X to the G-set $G \times X$, with G acting on the left. Prove that this adjunction is monadic by appealing to the monadicity theorem.

By Beck's monadicity theorem, a functor $U : \mathsf{D} \to \mathsf{C}$ is monadic if and only if U has a left adjoint and U creates coequalizers of U-split pairs. An equivalent statement is:

A functor $U : \mathsf{D} \to \mathsf{C}$ is monadic iff:

1. U has a left adjoint;
2. D has coequalizers of U-split pairs;
3. U preserves coequalizers of U-split pairs; and
4. U reflects isomorphisms.

To this end, the four conditions above must be shown for $U : \mathsf{Set}^{\mathsf{B}G} \to \mathsf{Set}$.

1. First, it is given that U has a left adjoint $F : \mathsf{Set} \to \mathsf{Set}^{\mathsf{B}G}$ given by $X \mapsto G \times X$ (G acts on the left).
2. $\mathsf{Set}^{\mathsf{B}G}$ has coequalizers of U-split pairs:
   Actually, we claim that $\mathsf{Set}^{\mathsf{B}G}$ has arbitrary coequalizers.
   For any diagram $D : \mathsf{J} \to \mathsf{Set}^{\mathsf{B}G}$, $\operatorname{colim}_{\mathsf{J}} D$ is the set of G-orbits (Ex. 3.5.i). By Theorem 3.4.12, the colimit of any small diagram $D : \mathsf{J} \to \mathsf{Set}^{\mathsf{B}G}$ can be expressed as a coequalizer of a pair of maps between coproducts. i.e., the following: if $f, g : A \rightrightarrows B$ is a pair of G-maps, their coequalizer is the canonical projection map
   $$\pi : B \twoheadrightarrow B/\sim$$
   where $\sim$ is the equivalence relation defining the G-orbits; $x \sim y$ iff $\exists g \in G$ such that $g \cdot x = y$. π equalizes f and g and satisfies the universal property of coequalizers, i.e.,
   $$\begin{array}{ccccc} A & \overset{f}{\underset{g}{\rightrightarrows}} & B & \twoheadrightarrow & B/\sim \\ & & {\scriptstyle\forall\gamma}\downarrow & \swarrow{\scriptstyle\exists!} & \\ & & C & & \end{array}$$
   As $\mathsf{Set}^{\mathsf{B}G}$ has arbitrary coequalizers, it has coequalizers of U-split pairs.
3. U preserves coequalizers of U-split pairs:
   Suppose f, g is a U-split pair. Their coequalizer in Set is the quotient $UB/\sim'$ where $\sim'$ is th equivalence relation $b_1 \sim' b_2$ iff $\exists\ a \in A$ such that $f(a) = b_1$ and $g(a) = b_2$. Denote this by p:
   $$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{p}{\twoheadrightarrow} B/\sim'$$
   To show that U preserves coequalizers of U-split pairs, it remains to show that
   $$B/\sim' \cong U(B/\sim)$$
4. U reflects isomorphisms:
   Suppose $f : A \to B$ is a morphism (G-map) in $\mathsf{Set}^{\mathsf{B}G}$ such that $U(f)$ is an isomorphism in Set. This implies that f is a bijection of sets. As f is a G-map by assumption, and a bijective G-map is an isomorphism in $\mathsf{Set}^{\mathsf{B}G}$, then $f \in \mathsf{Set}^{\mathsf{B}G}$ is an isomorphism. Hence, U reflects isomorphisms.

By Beck's monadicity theorem, then the adjunction is monadic.

□

Exercise 5.5.v. Generalizing Exercise 5.5.iv, for any small category J and any cocomplete category C, the forgetful functor $\mathsf{C}^{\mathsf{J}} \to \mathsf{C}^{\mathrm{ob}\mathsf{J}}$ admits a left adjoint $\mathrm{Lan} : \mathsf{C}^{\mathrm{ob}\mathsf{J}} \to \mathsf{C}^{\mathsf{J}}$ that sends a functor $F \in \mathsf{C}^{\mathrm{ob}\mathsf{J}}$ to the functor $\mathrm{Lan}F \in \mathsf{C}^{\mathsf{J}}$ defined by

$$\mathrm{Lan}F(j) = \coprod_{x \in \mathsf{J}} \coprod_{\mathsf{J}(x,j)} Fx$$

(i) Define $\mathrm{Lan}F$ on morphisms in J.

Suppose $f : a \to b$ is a morphism in J. Define $\mathrm{Lan}F(f) : \mathrm{Lan}F(a) \to \mathrm{Lan}F(b)$ by

$$\mathrm{Lan}F(f) : \coprod_{x \in \mathsf{J}} \coprod_{\mathsf{J}(x,a)} Fx \to \coprod_{y \in \mathsf{J}} \coprod_{\mathsf{J}(y,b)} Fy$$

where $Fx \mapsto Fy$ according to f by post-composition.

□

(ii) Define Lan on morphisms in $\mathsf{C}^{\mathrm{ob}\mathsf{J}}$.

Suppose $\alpha : F \Rightarrow G$ is a morphism in $\mathsf{C}^{\mathrm{ob}\mathsf{J}}$ where $F, G : \mathrm{ob}\mathsf{J} \to \mathsf{C}$ and the components are given by $\alpha_j : F(j) \to G(j)$ for each $j \in \mathrm{ob}\mathsf{J}$. Then, $\mathrm{Lan}\alpha = \alpha^*$ where

$$\alpha^* : \mathrm{Lan}F \Rightarrow \mathrm{Lan}G$$
$$\alpha^*_j : \mathrm{Lan}F(j) \to LanG(j)$$
$$\coprod_{x \in \mathsf{J}} \coprod_{\mathsf{J}(x,j)} Fx \mapsto \coprod_{x \in \mathsf{J}} \coprod_{\mathsf{J}(x,j)} Gx \circ \alpha_x$$

(iii) Use the Yoneda lemma to show that Lan is left adjoint to the forgetful (restriction) functor $\mathsf{C}^{\mathsf{J}} \to \mathsf{C}^{\mathrm{ob}\mathsf{J}}$.

(iv) Prove that this adjunction is monadic by appealing to the monadicity theorem.

Exercise 5.5.vii. Consider the Kleisli category Set_T for a monad T acting on Set and choose a skeleton $\mathsf{N} \xrightarrow{\simeq} \mathsf{Fin} \hookrightarrow \mathsf{Set}$ for the full subcategory of finite sets. Let L be the opposite of the full subcategory of the Kleisli category spanned by the objects $0, 1, 2, \ldots$ in N, so that there is an identity-on-objects functor:

$$\begin{array}{ccccc} \mathsf{N}^{\mathrm{op}} & & \overset{I}{\dashrightarrow} & & \mathsf{L} \\ {\scriptstyle \simeq}\downarrow & & & & \downarrow \\ \mathsf{Fin}^{\mathrm{op}} & \rightarrowtail & \mathsf{Set}^{\mathrm{op}} & \xrightarrow{F_T} & \mathsf{Set}_T^{\mathrm{op}} \end{array}$$

(i) Show that the categories N^{op} and L have strictly associative finite products that are preserved by the functor $I : \mathsf{N}^{\mathrm{op}} \to \mathsf{L}$.

– Strictly associative?

When the monad T is finitary, the functor $I : \mathsf{N}^{\mathrm{op}} \to \mathsf{L}$ defines its associated **Lawvere theory**. Because the objects in the category L are all iterated finite products of the object 1, the essential data in the Lawvere theory is the set $\mathsf{L}(n, 1)$ of "n-ary operations." It is possible to recover the category of T-algebras, that is, the category of models for this algebraic theory, from this data. A **model** of the Lawvere theory in Set is a finite product preserving functor $\mathsf{L} \to \mathsf{Set}$. A morphism between models is a natural transformation.

(ii) Define a functor from the category of T-algebras to the category of models for the Lawvere theory $I : \mathsf{N}^{\text{op}} \to \mathsf{L}$.

To define a functor $\mathsf{C}^T \to \text{mod}_{\mathsf{L}}$ where mod_{L} is the category of models for the Lawvere theory L, using exercise 5.2.vii, define:

$$T\text{-algebra} \mapsto \text{underlying set functor } U|_T : \mathsf{L} \to \mathsf{Set}$$

□