Chapter 4: Adjunctions

Section 4.1. Adjoint functors

Exercise 4.1.i. Show that functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ and bijections $\mathsf{D}(Fc, d) \cong \mathsf{C}(c, Gd)$ for each $c \in \mathsf{C}$ and $d \in \mathsf{D}$ define an adjunction if and only if these bijections induce a bijection between commutative squares:

$$\begin{array}{ccc} Fc & \xrightarrow{f^\sharp} & d \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle k} \\ Fc' & \xrightarrow[g^\sharp]{} & d' \end{array} \quad \leftrightsquigarrow \quad \begin{array}{ccc} c & \xrightarrow{f^\flat} & Gd \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Gk} \\ c' & \xrightarrow[g^\flat]{} & Gd' \end{array}$$

(i.e., prove Lemma 4.1.3.)

$\Rightarrow$: Suppose $\mathsf{D}(Fc, d) \cong \mathsf{C}(c, Gd)$ is natural in both variables for all $c \in \mathsf{C}, d \in \mathsf{D}$.

$\rightsquigarrow$: Suppose

$$\begin{array}{ccc} Fc & \xrightarrow{f^\sharp} & d \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle k} \\ Fc' & \xrightarrow[g^\sharp]{} & d' \end{array}$$

commutes in D. By naturality in D,

$$(kf^\sharp)^\flat = Gkf^\flat$$

Also, by naturality in C,

$$(g^\sharp Fh)^\flat = g^\flat h$$

So we have that commutes in C:

$$\begin{array}{ccc} c & \xrightarrow{f^\flat} & Gd \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Gk} \\ c' & \xrightarrow[g^\flat]{} & Gd' \end{array}$$

$\leftsquigarrow$: If the left-hand square commutes, going backwards in the above proof yields that the right-hand square commutes.

$\Leftarrow$: On the other hand, suppose that $\begin{array}{ccc} Fc & \xrightarrow{f^\sharp} & d \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle k} \\ Fc' & \xrightarrow[g^\sharp]{} & d' \end{array}$ commutes in D if and only if $\begin{array}{ccc} c & \xrightarrow{f^\flat} & Gd \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Gk} \\ c' & \xrightarrow[g^\flat]{} & Gd' \end{array}$ commutes in C.

To show that $\mathsf{D}(Fc, d) \cong \mathsf{C}(c, Gd)$ is natural in both variables, it suffices to show that

$$Gkf^\flat = (kf^\sharp)^\flat \text{ in } \mathsf{C}$$
$$f^\flat h = (f^\sharp Fh)^\flat \text{ in } \mathsf{C}$$

But this is the statement of the commutative squares (just applying $\flat$ and $\sharp$ and moving back and forth).

$\square$

Exercise 4.1.ii. Define left and right adjoints to

(i) ob : $\mathsf{Cat} \to \mathsf{Set}$,

ob : $\mathsf{Cat} \to \mathsf{Set}$ is defined by sending a category C to its set of objects and a functor $F : \mathsf{C} \to \mathsf{D}$ to its underlying set map ob$\mathsf{C} \to$ ob$\mathsf{D}$.
Left adjoint to ob: The functor Di : $\mathsf{Set} \to \mathsf{Cat}$ where:

$$\text{Di} : X \mapsto \text{ discrete category on } X$$

i.e., the category where objects are elements of X and morphisms are only identity morphisms.
Right adjoint to ob: The functor $G : \mathsf{Set} \to \mathsf{Cat}$ where:

$$G : X \mapsto \text{ category } \mathsf{C}_X \text{ with } \text{ob}\mathsf{C}_X = X$$

and C_X has exactly one arrow in each hom-set.

(ii) Vert : $\mathsf{Graph} \to \mathsf{Set}$, and

Vert : $\mathsf{Graph} \to \mathsf{Set}$ is defined by sending a graph Γ to its set of vertices and a graph morphism $\varphi : \Gamma \to \Psi$ to its underlying set map of vertices, $V_\Gamma \to V_\Psi$.
Left adjoint to Vert: The functor F : $\mathsf{Set} \to \mathsf{Graph}$ where:

$$F : X \mapsto \Gamma_X$$

where Γ_X is the graph with vertices elements of X and set maps are sent to the graph morphism defined on the vertices of each graph.
Right adjoint to Vert: The functor $G : \mathsf{Set} \to \mathsf{Graph}$ where:

$$G : X \mapsto \Gamma'_X$$

where Γ'_X is the graph with vertices X and one edge between every pair of vertices (the complete graph on $|X|$ vertices).

(iii) Vert : $\mathsf{DirGraph} \to \mathsf{Set}$. Vert : $\mathsf{DirGraph} \to \mathsf{Set}$ is defined by sending a directed graph Γ to its set of vertices and a directed graph morphism $\varphi : \Gamma \to \Psi$ to its underlying set map of vertices, $V_\Gamma \to V_\Psi$.
Left adjoint to Vert: The functor F : $\mathsf{Set} \to \mathsf{DirGraph}$ where:

$$F : X \mapsto \Gamma_X$$

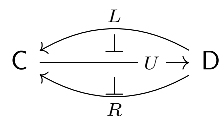
where Γ_X is the graph with vertices elements of X and set maps are sent to the graph morphism defined on the vertices of each graph.
Right adjoint to Vert: The functor $G : \mathsf{Set} \to \mathsf{DirGraph}$ where:

$$G : X \mapsto \Gamma'_X$$

where Γ'_X is the graph with vertices X and two edges between every pair of vertices (each Hom set has exactly 1 element, and morphisms are directed graph morphisms).

□

Exercise 4.1.iii. Show that any triple of adjoint functorsrt

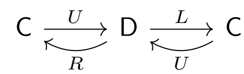


gives rise to a canonical adjunction $LU \dashv RU$ between the induced endofunctors of C.

Since $L \dashv U$, and $U \dashv R$, then for each $c \in \mathsf{C}$, $d \in \mathsf{D}$,

$$\mathsf{C}(Ld, c) \cong D(d, Uc)$$
$$\mathsf{D}(Uc, d) \cong \mathsf{C}(c, Rd)$$

and both isomorphisms are natural in both variables.

Want: natural isomorphism $\mathsf{C}(LUc', c) \cong \mathsf{C}(c', RUc)$ for each $c, c' \in \mathsf{C}$. We have that 

Denote the adjunctions $L \dashv U$ and $U \dashv R$ as follows.

$$\frac{\dfrac{LUc' \to c}{Uc' \to Uc}}{c' \to RUc}$$

Thus, by the first and last lines of the above visual, $LU \vdash RU$.

$\square$

Section 4.2. The unit and counit as universal arrows

Exercise 4.2.i. Prove that any pair of adjoint functors $F : \mathsf{C} \to \mathsf{D}$, $G : \mathsf{D} \to \mathsf{C}$ restrict to define an equivalence between the full subcategories spanned by those objects $c \in \mathsf{C}$ and $d \in \mathsf{D}$ for which the components of the unit η_c and of the counit ϵ_c, respectively, are isomorphisms.

Restrict F and G to the full subcategories of C and D respectively spanned by those $c \in \mathsf{C}, d \in \mathsf{D}$ such that η_c and ϵ_d are isomorphisms. For ease of notation, denote these as F and G. Then, F is fully faithful if and only if

$$\mathsf{C}(x, y) \to \mathsf{D}(Fx, Fy)$$

is an isomorphism in Set. Observe that

$$\mathsf{C}(x, y) \longrightarrow \mathsf{D}(Fx, Fy) \longrightarrow \mathsf{C}(x, GFy)$$

is naturalin x. By adjunction (and the Yoneda lemma), we know that

$$\mathsf{D}(Fx, Fy) \longrightarrow \mathsf{C}(x, GFy)$$

is an isomorphism.

Now, $\mathsf{C}(x, y) \to \mathsf{C}(x, GFy)$ is also an isomorphism:

$$(f : x \mapsto y) \longmapsto \eta_y \circ f \in \mathsf{C}(x, GFy)$$

As η_y is an isomorphism, then this map is injective and surjective. Hence,this is an isomorphism, which yields that $\mathsf{C}(x, y) \to \mathsf{D}(Fx, Fy)$ is an isomorphism. So F is fully faithful.

Similarly, G is fully faithful iff

$$\mathsf{D}(x', y') \to \mathsf{C}(Gx', Gy')$$

is an isomorphism in Set. Observe that

$$\mathsf{D}(x', y') \longrightarrow \mathsf{C}(Gx', Gy') \longrightarrow \mathsf{D}(FGx', y')$$

is determined by $\epsilon_{x'}$ so that $\mathsf{D}(x', y') \to \mathsf{D}(FGx', y')$ is given by $g \mapsto g \circ \epsilon_{x'}$.

F is essentially surjective: Let $d \in \mathsf{D}$. Take $c := Gd \in \mathsf{C}$. Since $\epsilon_d : FGd \to d$ is an isomorphism, then $Fc \cong d$ and hence F is essentially surjective.

That is, F and G are fully faithful and essentially surjective, and hence these restrictions define an equivalence.

□

Exercise 4.2.iii. Pick your favourite forgetful functor from Example 4.1.10 and prove that it is a right adjoint by defining its left adjoint, the unit, and the counit, and demonstrating that the triangle identities hold.

From example 4.1.10(iv): consider the forgetful functor $U : \mathsf{Ab} \to \mathsf{Set}$. The "free" construction $F : \mathsf{Set} \to \mathsf{Ab}$ where $X \mapsto \mathbb{Z}[X] := \oplus_X \mathbb{Z}$ is left adjoint to U.

Unit: The unit is given by $\eta : 1_{\mathsf{Set}} \Rightarrow UF$, where

$$\eta_A : A \to UFA$$

UFA is the underlying set of the abelian group on A, so this is the inclusion of generators.

Counit: The counit is given by $\epsilon : FU \Rightarrow 1_{\mathsf{Ab}}$, where

$$\epsilon_G : FUG \to G$$

Since $FX = \{\sum_{g \in G} c_g \cdot g \mid c_g \in \mathbb{Z}\}$, then $\sum_{g \in G} c_g \cdot g$ is the evaluation of the sum. This is what ϵ_G corresponds to.

Triangle identities: It is clear that $F\eta_A : FA \to FUFA$ has left inverse $\epsilon_{FA} : FUFA \to FA$ and $U\epsilon_G : UFUG \to UG$ has right inverse $\eta_{UG} : UG \to UFUG$.

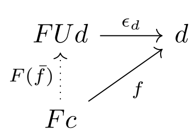
□

Exercise 4.2.iv. Each component of the counit of an adjunction is a terminal object in some category. What category?

Suppose $F \dashv G$, $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$, is an adjunction. Let $\epsilon : FG \Rightarrow 1_D$, $\epsilon_d : FGd \to d$ be the counit. Let $1_d : \mathbb{1} \to \mathsf{D}$ denote the constant functor at $d \in \mathsf{D}$.

Claim: ϵ_d is a terminal object of the comma category $F \downarrow 1_d$ (defined in Exercise 1.3.vi).

Let $(c \in \mathsf{C}, x \in \mathbb{1}, f : Fc \to 1_d x)$ be an arbitrary element of $F \downarrow 1_d$. Then, there exist unique $\bar{f} : c \to Ud$ such that this commutes:



i.e., there exists a unqiue morphism $(c \in \mathsf{C}, x \in \mathbb{1}, f : Fc \to 1_d x) \to (Gd \in \mathsf{C}, x \in \mathbb{1}, \epsilon_d : FGd \to 1_d x)$, so ϵ_d is the terminal object of this comma category.

□

Section 4.3. Contravariant and multivariable adjoint functors

Exercise 4.3.iii. Show that the contravariant power set functor $P : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ is mutually right adjoint to itself.

By definition, $P : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ is mutually right adjoint to itself if and only if there exists a natural isomorphism (for all $A \in \mathsf{Set}^{\mathrm{op}}$, $B \in \mathsf{Set}$).

$$\mathsf{Set}(B, PA) \cong \mathsf{Set}(A, PB)$$

Note the following adjunction for sets X, Y, Z:

$$\frac{X \xrightarrow{f} Z^Y}{X \times Y \underset{g}{\longrightarrow} Z}$$

This is given by $\overline{()}$ from top to bottom as:

$$\bar{f}(x, y) = f(x)(y)$$

and $\overset{*}{()}$ from bottom to top is given by:

$$g^*(x)(y) = g(x, y)$$

Then, clearly $- \times Y \dashv (-)^Y$.

Using this, we have the following sequence of adjunctions (notation as in Exercise 4.1.iii.).

$$\begin{array}{c} A \to P(B) \\ \hline A \to \Omega^B \\ \hline A \times B \to \Omega \\ \hline B \times A \to \Omega \\ \hline B \to \Omega^A \\ \hline B \to P(A) \end{array}$$

Hence, P is mutually right adjoint to itself.

□

Section 4.5. Adjunctions, limits, and colimits

Exercise 4.5.i. When does the unique functor $! : \mathsf{C} \to \mathbb{1}$ have a left adoint? When does it have a right adjoint?

The functor $! : \mathsf{C} \to \mathbb{1}$ has a left adjoint when C has an initial object. Denoting the initial object as s, the counit of the adjunction is $\epsilon : s \to c$.

Similarly, functor $! : \mathsf{C} \to \mathbb{1}$ has a right adjoint when C has a terminal object t. The unit of the adjunction is then $\eta : c \to t$.

□

Exercise 4.5.ii. Suppose the diagonal functor $\Delta : \mathsf{C} \to \mathsf{C}^{\mathsf{J}}$ admits both left and right adjoints. Describe the units and counits of these adjunctions.

The left adjoint to $\Delta : \mathsf{C} \to \mathsf{C}^J$ is the colimit object and the unit is the universal cone (in definition of colimit); and similarly the right adjoint to Δ is the limit object with counit the universal cone.

□

Section 4.6. Existence of adjoint functors

Exercise 4.6.ii. Use Theorem 4.6.3 to prove that the inclusion $\mathsf{Haus} \hookrightarrow \mathsf{Top}$ of the full subcategory of Hausdorff spaces into the category of all spaces has a left adjoint. The left adjoint carries a space to its "largest Hausdorff quotient." Conclude, by applying Proposition 4.5.15, that the category of Hausdorff spaces, as a reflective subcategory of a complete and cocomplete category, is cocomplete as well as complete.

First, it is clear that any product of Hausdorff spaces is Hausdorff and a subspace of a Hausdorff space is also Hausdorff. Thus, Haus is complete and $U : \mathsf{Haus} \hookrightarrow \mathsf{Top}$ is a continuous functor.

To show that U has a left adjoint, it remains to show that the solution set condition holds (by the *Adjoint Functor Theorem* 4.6.3). To this end, let $X \in \mathsf{Top}$ be an arbitrary topological space. Let $f : X \to Y$ be a continuous map where $Y \in \mathsf{Haus}$. Then, clearly the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & \nearrow_{\iota} & \\ f(X) & & \end{array}$$

As $f(X) \subset Y$ is a subspace, then it is also Hausdorff, so $f(X) \in \mathsf{Haus}$. Also, f factors through f along the inclusion morphism $\iota : f(X) \hookrightarrow Y$. That is, the solution set condition holds and thus U has a left adjoint.

By Proposition 4.5.15, as Haus is a reflective subcategory (U admits a left adjoint), then Haus admits all small colimits, so it is cocomplete.

$\square$