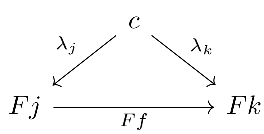
# Chapter 3: Limits and Colimits

## Section 3.1. Limits and colimits as universal cones

Exercise 3.1.v. Consider a diagram $F : \mathsf{J} \to \mathsf{P}$ valued in a poset $(\mathsf{P}, \leq)$. Use order-theoretic language to characterize the limit and the colimit.

By definition 3.1.6, for a diagram $F : \mathsf{J} \to \mathsf{P}$, a limit is a terminal object in the category of cones over $F$, i.e., in $\int \mathrm{Cone}(-, F)$. An object of $\int \mathrm{Cone}(-, F)$ is a cone over $F$ with any summit. A cone over $F : \mathsf{J} \to \mathsf{P}$ with summit $c \in \mathsf{P}$ is a natural transformation $\lambda : c \Rightarrow F$. The components $(\lambda_j : c \to F_j)_{j \in \mathsf{J}}$ are called the legs. i.e., it is the collection of morphisms $\lambda_j : c \to F_j$ such that for all morphisms $f : j \to k \in \mathsf{J}$, this commutes in $\mathsf{P}$:




*Claim*: $\lim F = \inf\{Fj \mid j \in \mathsf{J}\}$.

As $\inf\{Fj \mid Fj \in \mathsf{J}\} \leq F_j$ for all $j$, then this is a cone. Also, if $p \leq Fj$ for all $j$, then $p \leq \lim F$ by definition of infimum. That is, there exists a unique morphism $f : p \to \lim F$. So $\lim F$ is the terminal object of $\int \mathrm{Cone}(-, F)$ as required.

*Claim*: $\mathrm{colim} F = \sup\{Fj \mid j \in \mathsf{J}\}$.

Since $\sup\{Fj \mid j \in \mathsf{J}\} \geq Fj$ for all $j \in \mathsf{J}$, then this is a cone under $F$ by definition. Also, if $q \geq F_j$ for all $j$ then $q \geq \mathrm{colim} F$ by definition of supremum, so there exists a unique morphsim $g : \mathrm{colim} F \to q$ That is, this is the initial object of $\int \mathrm{Cone}(F, -)$, as required.

□

Exercise 3.1.vi. Prove that if

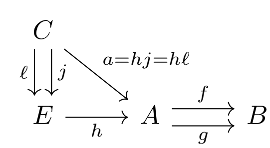
$$E \xrightarrow{h} A \overset{f}{\underset{g}{\rightrightarrows}} B$$

is an equalizer diagram, then $h$ is a monomorphism.

Let $j, \ell : C \to E$ be morphisms such that

$$hj = h\ell$$

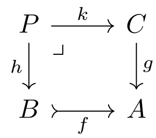
Let $a = hj = h\ell$. Then, the following commutes.




Since $E \xrightarrow{h} A \overset{f}{\underset{g}{\rightrightarrows}} B$ is an equalizer diagram, by the universal property, $\exists!\ k : C \to E$ such that $a = hk$. By uniqueness and the above diagram, then $k = j = \ell$, so $h$ is a monomorphism.

□

Exercise 3.1.vii. Prove that if

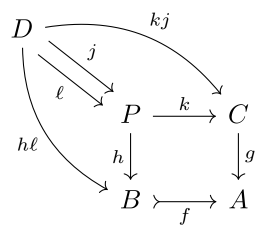



is a pullback square and $f$ is a monomorphism, then $k$ is a monomorphism.

Suppose $f$ is a monomorphism. Let $j, \ell : D \to P$ be such that

$$kj = k\ell$$

By hypothesis, this commutes.




Then, $gkj = gk\ell$. Since the diagram commutes, then

$$fhj = fh\ell$$

Since $f$ is a monomorphism, then

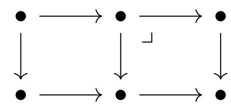
$$hj = h\ell$$

That is, $kj = k\ell$ and $hj = h\ell$. By the universal property of the pullback $P$, then

$$j = \ell$$
$$\Rightarrow \quad k \text{ is a monomorphism.}$$

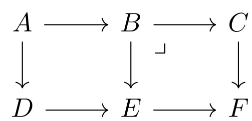
□

Exercise 3.1.viii. Consider a commutative rectangle

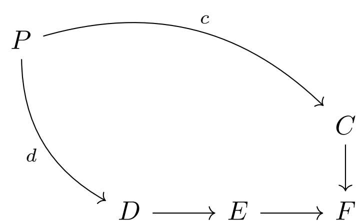


whose right-hand square is a pullback. Show that the left-hand square is a pullback if and only if the composite rectangle is a pullback.

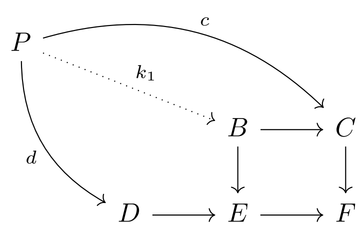
Label the diagram as follows.



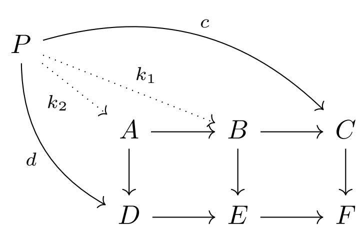

⇒: Suppose the left-hand square is a pullback. To show that the composite rectangle is a pullback, suppose the following diagram commutes:




Since the right-hand square is a pullback, there exists a unique map $k_1 : P \to B$ such that this commutes:



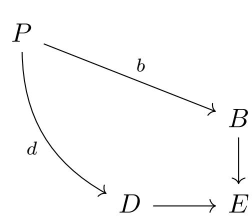

Since the left-hand square is a pullback, then there exists a unique $k_2 : P \to A$ such that the following commutes.



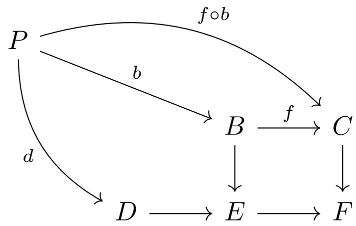

Therefore, the composite rectangle is a pullback.

⇐: On the other hand, suppose the composite rectangle is a pullback.

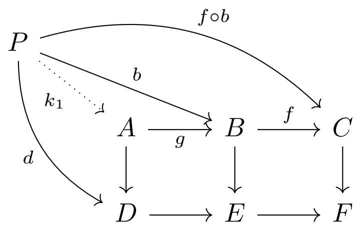
To show that the left-hand square is a pullback, suppose the following diagram commutes.



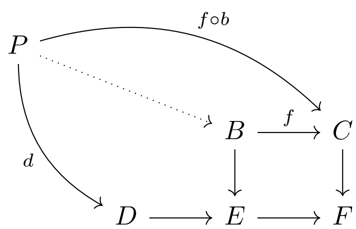
Then, this also commutes




Since the composite rectangle is a pullback, $\exists!\ k_1 : P \to A$ such that




Hence, if $g \circ k_1 = b$, then it follows that the left-hand square is also a pullback. To this end, note that they both "fit" into the commutative diagram given by:




Since the right-hand square is a pullback, there exists exactly one such morphism (by uniqueness) and hence

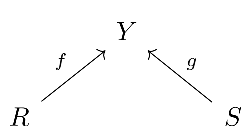
$$g \circ k_1 = b$$

Thus, the left-hand square is a pullback.

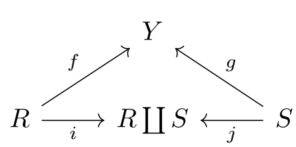
□

Exercise 3.1.xiii. What is the coproduct in the category of commutative rings?

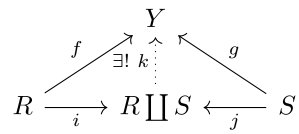
A coproduct $\coprod_{j \in \mathsf{J}} A_j$ is the colimit of a diagram $(A_j)_{j \in \mathsf{J}}$ indexed by a discrete category $\mathsf{J}$. A cone under such a diagram is a $\mathsf{J}$-indexed family of ring homomorphisms $(\lambda_j : Fj \to c)_{j \in \mathsf{J}}$ with no further constraints. Suppose $R, S \in \mathsf{CRing}$ are rings with nadir a ring $Y \in \mathsf{CRing}$. That is, there are ring homomorphisms $f : R \to Y$ and $g : S \to Y$.




The legs of this cone are (inclusion) ring homomorphisms $i, j$ such that $i : R \to R \coprod S, j : S \to R \coprod S$ so that this commutes:



By the coproduct being being the initial object, there exists a unique ring homomorphism $k : R \coprod S \to Y$ such that




This is exactly the universal property of $R \otimes_{\mathbb{Z}} S$, so $R \coprod S = R \otimes_{\mathbb{Z}} S$ (tensor product of the underlying abelian groups of $R$ and $S$).

$\square$

## Section 3.2. Limits in the category of sets

Exercise 3.2.v. Show that for any small category $\mathsf{J}$, any locally small category $\mathsf{C}$, and any pair of parallel functors $F, G : \mathsf{J} \rightrightarrows \mathsf{C}$, there is an equalizer diagram

$$\begin{array}{ccccc} & & \mathsf{C}(Fj', Gj') & \xrightarrow{Ff^*} & \mathsf{C}(Fj, Gj') \\ & & \uparrow{\scriptstyle \pi_{j'}} & & \uparrow{\scriptstyle \pi_f} \\ \mathrm{Hom}(F,G) & \rightarrowtail & \prod\limits_{j \in \mathrm{ob}\mathsf{J}} \mathsf{C}(Fj, Gj) & \rightrightarrows & \prod\limits_{f : j \to j' \in \mathrm{mor}\mathsf{J}} \mathsf{C}(Fj, Gj') \\ & & \downarrow{\scriptstyle \pi_j} & & \downarrow{\scriptstyle \pi_f} \\ & & \mathsf{C}(Fj, Gj) & \xrightarrow[Gf_*]{} & \mathsf{C}(Fj, Gj') \end{array}$$

Note: this construction gives a second formula for $\mathrm{Hom}(F,G)$ as a limit in $\mathsf{Set}$.

First note that by definition, $Ff^* = - \cdot Ff$ and $Gf_* = Gf \cdot -$. Now, let $h : \mathrm{Hom}(F,G) \rightarrowtail \prod\limits_{j \in \mathrm{ob}\mathsf{J}} \mathsf{C}(Fj, Gj)$ denote the monomorphism such that $(\alpha : F \Rightarrow G) \overset{h}{\longmapsto} \prod\limits_{j \in \mathsf{J}} \alpha_j$ where $(\alpha_j : F_j \to G_j)_j$ are the components of $\alpha$. Applying $\pi_{j'}$ followed by $Ff^*$ yields $\alpha_{j'} \circ Ff$ for each component $\alpha_{j'}$. Hence, the top arrow in the double arrows is $F^*$ (pre-composition by $F$) and analogously the bottom arrow is $G_*$. i.e., we have

$$\begin{array}{ccccc} & & \mathsf{C}(Fj', Gj') & \xrightarrow{-\cdot Ff} & \mathsf{C}(Fj, Gj') \\ & & \uparrow{\scriptstyle \pi_{j'}} & & \uparrow{\scriptstyle \pi_f} \\ \mathrm{Hom}(F,G) & \overset{h}{\rightarrowtail} & \prod\limits_{j \in \mathrm{ob}\mathsf{J}} \mathsf{C}(Fj, Gj) & \underset{G_*}{\overset{F^*}{\rightrightarrows}} & \prod\limits_{f : j \to j' \in \mathrm{mor}\mathsf{J}} \mathsf{C}(Fj, Gj') \\ & & \downarrow{\scriptstyle \pi_j} & & \downarrow{\scriptstyle \pi_f} \\ & & \mathsf{C}(Fj, Gj) & \xrightarrow[Gf\cdot -]{} & \mathsf{C}(Fj, Gj') \end{array}$$

That is, we have the following for $\alpha \in \mathrm{Hom}(F,G)$:

$$\alpha \overset{h}{\longmapsto} (\alpha_j)_{j \in \mathsf{J}} \overset{F^*}{\longmapsto} (\alpha_j \circ Ff)_{f \in \mathrm{mor}\mathsf{J}}$$

$$\alpha \overset{h}{\longmapsto} (\alpha_j)_{j \in \mathsf{J}} \overset{G_*}{\longmapsto} (Gf \circ \alpha_j)_{f \in \mathrm{mor}\mathsf{J}}$$

To show that $\mathrm{Hom}(F,G)$ is an equalizer, let $a : D \to \prod\limits_{j \in \mathrm{ob}\mathsf{J}} \mathsf{C}(Fj, Gj) \in \mathsf{Set}$ be arbitrary. i.e.,

$$\begin{array}{ccccc} D & & & & \\ & \searrow^{a} & & & \\ \mathrm{Hom}(F,G) & \underset{h}{\rightarrowtail} & \prod\limits_{j \in \mathrm{ob}\mathsf{J}} \mathsf{C}(Fj, Gj) & \underset{G_*}{\overset{F^*}{\rightrightarrows}} & \prod\limits_{f : j \to j' \in \mathrm{mor}\mathsf{J}} \mathsf{C}(Fj, Gj') \end{array}$$

That is, for each $d \in D$,

$$a(d) = (g_j^d)_{j \in \mathrm{ob}\mathsf{J}}$$

where $g_j^d : F_j \to G_j \in \mathsf{C}$ for each $j$ is completely characterized by $a$.

Now, define $m : D \to \mathrm{Hom}(F, G)$ by

$$m : d \mapsto \alpha^d : F \Rightarrow G$$
$$\text{where } g_j^d = \alpha_j^d : F_j \to G_j \text{ is the } j^{th} \text{ summand of } a(d)$$

$$\begin{array}{ccccc} D & & & & \\ {\scriptstyle m}\,\downarrow & \searrow^{a} & & & \\ \mathrm{Hom}(F,G) & \xrightarrow[h]{} & \prod_{j \in \mathrm{ob}\mathsf{J}} \mathsf{C}(Fj, Gj) & \underset{G_*}{\overset{F^*}{\rightrightarrows}} & \prod_{f : j \to j' \in \mathrm{mor}\mathsf{J}} \mathsf{C}(Fj, Gj') \end{array}$$

This defines a natural transformation since by the commutativity of the diagram,

$$Ff^* \pi_{j'} a = Gf_* \pi_j a$$

and by construction this means that $hm = a$. This is also unique since it is completely characterized by $a$. As the universal property of the equalizer holds, then $\mathrm{Hom}(F, G)$ is an equalizer in the diagram.

□

## Section 3.3. Preservation, reflection, and creation of limits and colimits

Exercise 3.3.ii. Prove Lemma 3.3.5, that a full and faithful functor reflects both limits and colimits.

Suppose $F : \mathsf{C} \to \mathsf{D}$ is a fully faithful functor. Given a diagram $D : \mathsf{J} \to \mathsf{C}$, then we have the composite $\mathsf{J} \xrightarrow{D} \mathsf{C} \xrightarrow{F} \mathsf{D}$. Suppose $(FL, F\lambda)$ is the limit in $\mathsf{D}$ of $FD$. Our candidate limit in $\mathsf{C}$ is $(L, \lambda)$. It must be shown that this is indeed a limit in $\mathsf{C}$.

To this end, let $(X, \alpha)$ be a cone over $D$. Then, $(FX, F\alpha)$ is a cone over $FD$. Then, there exists a unique map of cones $f : FX \to FL$ from $(FX, F\alpha)$ to $(FL, F\lambda)$. As $F$ is fully faihtful, then there exists a unique map of cones $\bar{f} : X \to L$ such that $F\bar{f} = f$. This map $\bar{f}$ is the unique map of cones, showing that $(L, \lambda)$ is indeed a limit in $\mathsf{C}$. Thus, a full and faithful functor reflects limits (and colimits).

□

Exercise 3.3.v. Show that the forgetful functors $U : \mathsf{Set}_* \to \mathsf{Set}$ and $U : \mathsf{Top}_* \to \mathsf{Top}$ fail to preserve coproducts and explain why this result demonstrates that the connectedness hypothesis in Proposition 3.3.8(ii) is necessary.

$U : \mathsf{Set}_* \to \mathsf{Set}$ does not preserve coproducts: in $\mathsf{Set}_*$, a coproduct of $A$ and $B$ is $(A \coprod b, c)$. The basepoint $c$ is given by identifying the basepoint $a \in A$ with the basepoint $b \in B$. On the other hand, in $\mathsf{Set}$, the coproduct is $A \coprod B$, but

$$(A \coprod B, c) \ncong A \coprod B$$

in general.

Also, $U : \mathsf{Top}_* \to \mathsf{Top}$ fails to preserve coproducts: a coproduct in $\mathsf{Top}_*$ is the wedge sum, i.e.,

$$X \vee Y = X \coprod Y / \sim \qquad \text{where } x_0 \sim y_0$$

where $x_0 \in X$, $y_0 \in Y$ are the respective basepoints. On the other hand, in $\mathsf{Top}$, the coproduct of $X$ and $Y$ is $X \coprod Y$ (disjoint union). But, clearly

$$X \vee Y \ncong X \coprod Y$$

The connectedness hypothesis in Proposition 3.3.8(ii) is necessary:

A coproduct $\coprod_{j \in \mathsf{J}} A_j$ is the colimit of a diagram indexed by a *discrete* category $\mathsf{J}$. But a discrete category is <u>not connected</u> (it only has identity morphisms). Thus, the coproduct is not a connected colimit, and so this hypothesis is needed for 3.3.8(ii) to hold.

□

## Section 3.4. The representable nature of limits and colimits

Exercise 3.4.ii. Explain in your own words why the Yoneda embedding $\mathsf{C} \hookrightarrow \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ preserves and reflects but does not create limits.

By the Yoneda Lemma, the Yoneda embedding $y : \mathsf{C} \hookrightarrow \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ is full and faitfhful:

$$\begin{array}{ccc} c & \longmapsto & \mathsf{C}(-, c) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} \\ d & \longmapsto & \mathsf{C}(-, d) \end{array}$$

$y$ is fully faithful so it reflects limits by Lemma 3.3.5. $y$ preserves limits since by Theorem 3.4.2, for any $F : \mathsf{J} \to \mathsf{C}$ whose limit exists: $\mathsf{C}(X, \lim_{\mathsf{J}} F) \cong \lim_{\mathsf{J}} \mathsf{C}(X, F-)$. And $\mathsf{C}(-, \lim F)$ is the limit of the composite diagram $\mathsf{J} \xrightarrow{F} C \xhookrightarrow{y} \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$, so $y$ preserves all limits that exist in $\mathsf{C}$; but clearly $y$ is not an equivalence in general and hence cannot create all limits.

## Section 3.5. Complete and cocomplete categories

Exercise 3.5.i. Let $G$ be a group regarded as 1-object category $\mathsf{B}G$. Describe the colimit of a diagram $\mathsf{B}G \to \mathsf{Set}$ in group-theoretic terms, as was done for the limit in Example 3.2.12.

To determine the colimit of a left $G$-set $X : \mathsf{B}G \to \mathsf{Set}$, define $\sim$ by $x \sim y$ iff there exists $g \in G$ such that $gx = y$. This is an equivalence relation since $ex = x$ (reflexive); if $gx = y$ then $g^{-1}y = x$ (symmetric); and if $gx = y, hy = z$, then $hgx = z$ (transitive). The colimit of $X$ is then the quotient $X/\sim =$ set of $G$-orbits.

□

Exercise 3.5.v. Describe the limits and colimits in the poset of natural numbers with the order relation $k \leq n$ if and only if $k$ divides $n$.

All diagrams in a poset commute, so whether or not there are any morphisms in a diagram is not relevant.

Let $F : \mathsf{J} \to \mathbb{N}$ be a diagram.

*Claim*: $\lim F = \gcd\{Fj \mid j \in \mathsf{J}\}$

First, $\lim F \leq Fj$ for all $j$, hence this is a cone. Also, if $p \leq Fj$ for all $j$, so that $p \mid Fj$ for all $j$, then $p \mid \gcd\{Fj \mid j \in \mathsf{J}\}$ by the greatest common divisor, so $p \leq \lim F$. Hence, this claim holds.

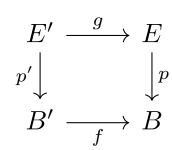
*Claim*: $\text{colim} F = \text{lcm}\{Fj \mid j \in \mathsf{J}\}$.

Similarly, $Fj \leq \text{colim}F$ for all $j$, hence this is a cone. Also, if $Fj \leq p$ for all $j$, so that $Fj \mid p$ for all $j$, then $\text{lcm}\{Fj \mid j \in \mathsf{J}\} \mid p$ by the least common multiple, so $\text{colim}F \leq p$. Hence, this claim holds.

□

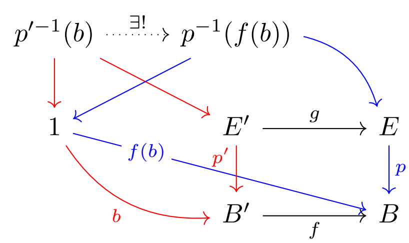
Exercise 3.5.vii. Following [**Gro58**], define a **fiber space** $p : E \to B$ to be a morphism in $\mathsf{Top}$. A map of fiber spaces is a commutative square. Thus, the category of fiber spaces is isomorphic to the diagram category $\mathsf{Top}^{\mathbb{2}}$. We are also interested in the non-full subcategory $\mathsf{Top}/B \subset \mathsf{Top}^{\mathbb{2}}$ of fiber spaces over $B$ and maps whose codomain component is the identity. Prove the following:

(i) A map




of fiber spaes induces a canonical map from the fiber over a point $b \in B'$ to the fiber over its image $f(b) \in B$.

The canonical map from the fiber over a point $b \in B'$ to the fiber over its image $f(b) \in B$ is given by the unique map determined by the following commutative diagram (pullback squares in red and blue):



□

(ii) The fiber of a product of fiber spaces is the product of the fibers.

It must be shown that the following holds:

$$\begin{array}{c} E \times E \\ \big\downarrow{\scriptstyle p\times p'} \\ B \times B' \end{array} \quad = \quad \begin{array}{c} E \\ \big\downarrow{\scriptstyle p} \\ B \end{array} \quad \times \quad \begin{array}{c} E' \\ \big\downarrow{\scriptstyle p'} \\ B' \end{array}$$

The fiber yields the following pullback:

$$\begin{array}{ccc} (p \times p')^{-1}(b, b') & \longrightarrow & E \times E' \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p\times p'} \\ 1 & \xrightarrow{(b,b')} & B \times B' \end{array}$$

Since products commute with pullbacks, then

$$(p \times p')^{-1}(b, b') \cong p^{-1}(b) \times p'^{-1}(b')$$

Thus, the fiber of a product of fiber spaces is the product of the fibers.

□

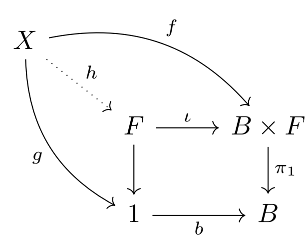
A projection $B \times F \to B$ defines a **trivial fiber space** over $B$, a definition that makes sense for any space $F$.

(iii) Show that the fiber of a trivial fiber space $B \times F \to B$ is isomorphic to $F$.

To show that the fiber of $B \times F \to B$ is isomorphic to $F$, it is equivalent to this being a pullback:

$$\begin{array}{ccc} F & \overset{x \mapsto (b,x)}{\underset{\iota}{\longrightarrow}} & B \times F \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\ 1 & \xrightarrow{b} & B \end{array}$$

The following holds for $X \in \mathsf{Top}$:




where $h(x) := \pi_2 \circ f(x)$. This commutes since $\iota h(x) = f(x)$ and $\pi_1 f(x) = b$ by commutativity of $\begin{array}{ccc} X & \longrightarrow & B \times F \\ \big\downarrow & & \big\downarrow \\ 1 & \longrightarrow & B \end{array}$. Hence, the aforementioned square is indeed a pullback square.

□

(iv) Characterize the isomorphisms in $\mathsf{Top}/B$ between two trivial fiber spaces (with a priori distinct fibers) over $B$.

Suppose we have two trivial fiber spaces over $B$ as follows.

$$\begin{array}{ccc} B \times F & \xrightarrow{\quad f \quad} & B \times G \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_1'} & \\ & B & \end{array}$$

<u>Claim</u>: $f = \mathrm{id}_B \times f'$ where $f' : F \to G$.
Let $(x, y) \in B \times F$. Then,

$$\begin{aligned} \pi_1(x, y) &= x \\ \pi_1'(x, y) = \pi_1'(f_1(x), f_2(x)) &= f_1(x) \\ \Rightarrow \quad f_1 &= \mathrm{id} \end{aligned}$$

Hence, $f' : F \to G$ is an isomorphism if there exists an isomorphism $f'' : G \to F$.

□

(v) Prove that the assignment of the set of continuous sections of a fiber space over $B$ defines a functor $\mathrm{Sect} : \mathsf{Top}/B \to \mathsf{Set}$.

The functor Sect is representable:

$$\mathsf{Top}/B \left( \begin{array}{c} B \\ \downarrow {\scriptstyle \mathrm{id}_B} \\ B \end{array}, - \right) : \mathsf{Top}/B \to \mathsf{Set}$$

That is, we have:

$$\begin{array}{ccccc} B & \xrightarrow{s} & E & \xrightarrow{f} & E' \\ & {\scriptstyle \mathrm{id}_B} \searrow & \downarrow {\scriptstyle p} & \swarrow {\scriptstyle p'} & \\ & & B & & \end{array}$$

so that

$$\mathsf{Top}/B(\mathrm{id}_B, p) \to \mathsf{Top}/B(\mathrm{id}_B, p')$$

is given by

$$s \longmapsto f \circ s$$

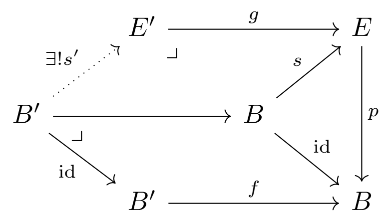
Hence, we have the following pullback square:

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ {\scriptstyle p'} \downarrow & \lrcorner & \downarrow {\scriptstyle p} \\ B' & \xrightarrow[f]{} & B \end{array}$$

Given $p$, define $\mathrm{Sect}(p) := \{ s : B \to E \mid ps = \mathrm{id} \}$. That is,

$$\mathrm{Sect} : (\mathsf{Top}^{\mathbb{2}}_{\mathrm{pb}})^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

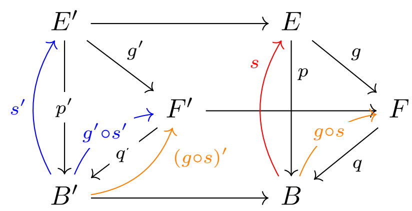
and given $s \in \mathrm{Sect}(p)$, $s' \in \mathrm{Sect}(p')$ must be defined by the following diagram (of pullback squares).




□

(vi) Consider the non-full subcategory $\mathsf{Top}^{\mathbf{2}}_{\mathrm{pb}}$ of fiber spaces in which the morphisms are the pullback squares. Prove that the assignment of the set of continuous sections to a fiber space defines a functor $\mathrm{Sect} : (\mathsf{Top}^{\mathbf{2}}_{\mathrm{pb}})^{\mathrm{op}} \to \mathsf{Set}$.

We have the following diagram with the horizontal direction being in $\mathsf{Top}^{\mathbf{2}}_{\mathrm{pb}}$ and the vertical direction in $\mathsf{Top}/B$:




This gives us that $(g \circ s)' = g' \circ s'$ thus Sect defines a functor $(\mathsf{Top}^{\mathbf{2}}_{\mathrm{pb}})^{\mathrm{op}} \to \mathsf{Set}$.

□