Chapter 2: Universal Properties, Representability, and the Yoneda Lemma

Section 2.1. Representable Functors

Exercise 2.1.i. For each of the three functors

$$\mathbb{1} \xrightarrow{0} \mathbb{2}$$
$$\mathbb{1} \xrightarrow{1} \mathbb{2}$$
$$\mathbb{1} \xleftarrow{!} \mathbb{2}$$

between the categories $\mathbb{1}$ and $\mathbb{2}$, describe the corresponding natural transformations between the covariant functors $\mathsf{Cat} \rightrightarrows \mathsf{Set}$ represented by the categories $\mathbb{1}$ and $\mathbb{2}$.

$\mathbb{1}$ is the category with one object (label it $*$) and its identity morphism. $\mathbb{2}$ is the category with two objects (label them 0 and 1), their identity morphisms, and one non-identity morphism usually labelled $0 \to 1$.

The covariant functor represented by $\mathbb{1}$ is $\mathrm{Hom}(\mathbb{1}, -) : \mathsf{Cat} \to \mathsf{Set}$, $\mathsf{C} \mapsto \mathrm{Hom}(\mathbb{1}, \mathsf{C})$. This covariant functor is naturally isomorphic to the objects of C, and the covariant functor represented by $\mathbb{2}$ is naturally isomorphic to the morphisms of a category C. i.e.,

$$\mathrm{Hom}(\mathbb{1}, \mathsf{C}) \cong \mathrm{Ob}\mathsf{C}$$
$$\mathrm{Hom}(\mathbb{2}, \mathsf{C}) \cong \mathrm{Mor}\mathsf{C}$$

This second isomorphism associates the morphisms in C with where the two objects of $\mathbb{2}$ are sent in C (the endpoints of the arrow).

Now, consider the functor $\mathbb{1} \xrightarrow{0} \mathbb{2}$ that sends the object $* \in \mathbb{1}$ to $0 \in \mathbb{2}$. This induces the natural transformation $\mathrm{Hom}(\mathbb{2}, \mathsf{C}) \to \mathrm{Hom}(\mathbb{1}, \mathsf{C})$ denoted by 0^* and defined by the following.

$$\mathrm{Mor}\mathsf{C} \cong \mathrm{Hom}(\mathbb{2}, \mathsf{C}) \xrightarrow{0^*} \mathrm{Hom}(\mathbb{1}, \mathsf{C}) \cong \mathrm{Ob}\mathsf{C}$$
$$(F : \mathbb{2} \to \mathsf{C}) \longmapsto 0^*F = F \circ 0$$

Similarly, consider the functor $\mathbb{1} \xrightarrow{1} \mathbb{2}$ that sends the object $* \in \mathbb{1}$ to $1 \in \mathbb{2}$. This induces the natural transformation $\mathrm{Hom}(\mathbb{2}, \mathsf{C}) \to \mathrm{Hom}(\mathbb{1}, \mathsf{C})$ denoted by 1^* and defined by the following.

$$\mathrm{Mor}\mathsf{C} \cong \mathrm{Hom}(\mathbb{2}, \mathsf{C}) \xrightarrow{1^*} \mathrm{Hom}(\mathbb{1}, \mathsf{C}) \cong \mathrm{Ob}\mathsf{C}$$
$$(F : \mathbb{2} \to \mathsf{C}) \longmapsto 1^*F = F \circ 1$$

Lastly, consider the functor $\mathbb{2} \xrightarrow{!} \mathbb{1}$ that sends the objects in $\mathbb{2}$ to the single object $* \in \mathbb{1}$ and all morphisms to the identity of $*$. This induces the natural transformation $\mathrm{Hom}(\mathbb{1}, \mathsf{C}) \to \mathrm{Hom}(\mathbb{2}, \mathsf{C})$ denoted by $!^*$ and defined by the following.

$$\mathrm{Ob}\mathsf{C} \cong \mathrm{Hom}(\mathbb{1}, \mathsf{C}) \xrightarrow{!^*} \mathrm{Hom}(\mathbb{2}, \mathsf{C}) \cong \mathrm{Mor}\mathsf{C}$$
$$x \mapsto 1_x$$

where $x \in \mathsf{C}$ is an object and 1_x is its identity morphism.

$\square$

Exercise 2.1.ii. Prove that if $F : \mathsf{C} \to \mathsf{Set}$ is representable, then F preserves monomorphisms, i.e., sends every monomorphism in C to an injective function. Use the contrapositive to find a covariant set-valued functor defined on your favourite concrete category that is not representable.

Suppose $F : \mathsf{C} \to \mathsf{Set}$ is representable. Then, there exists some $c \in \mathsf{C}$ such that

$$F \cong \mathsf{C}(c, -)$$

That is, there exists a natural transformation $\alpha : \mathsf{C} \Rightarrow \mathsf{Set}$ such that for all $x \in \mathsf{C}$, $\alpha_x : Fx \to \mathsf{C}(c, x)$ is an isomorphism and this commutes for all $f : x \to y \in \mathsf{C}$.

$$\begin{array}{ccc} Fx & \xrightarrow{\alpha_x} & \mathsf{C}(c,x) \\ {\scriptstyle Ff}\downarrow & \quad \alpha_y^{-1} \quad & \downarrow{\scriptstyle f_*} \\ Fy & \xrightarrow[\alpha_y]{} & \mathsf{C}(c,y) \end{array}$$

Now, let $f : x \to y \in \mathsf{C}$ be a monomorphism. Consider $Ff : Fx \to Fy \in \mathsf{Set}$. To show that Ff is injective, suppose that for some $x_1, x_2 \in Fx$,

$$Ff(x_1) = Ff(x_2)$$

By the above commutative diagram, then the following holds (α_y is an isomorphism),

$$Ff = \alpha_y^{-1} \circ f_* \circ \alpha_x$$

Hence, the following holds.

$$\begin{aligned} Ff(x_1) &= Ff(x_2) \\ \alpha_y^{-1} \circ f_* \circ \alpha_x(x_1) &= \alpha_y^{-1} \circ f_* \circ \alpha_x(x_2) \\ f_* \circ \alpha_x(x_1) &= f_* \circ \alpha_x(x_2) \end{aligned}$$

Since f_* is post-composition by f, then it follows that

$$f \circ \alpha_x(x_1) = f \circ \alpha_x(x_2)$$

Now, f is a monomorphism, and hence

$$\alpha_x(x_1) = \alpha_x(x_2)$$

As α_x is an isomorphism, then $x_1 = x_2$. Thus, Ff is injective so representable functors preserve monomorphisms.

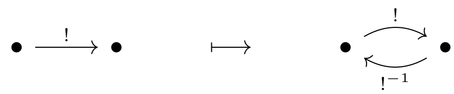
An example of a non-representable functor is given by an extension of exercise 1.6.iv. Recall the functor $F^{\text{op}} : \mathsf{Haus} \hookrightarrow \mathsf{Top}$ does not preserve monomorphisms. Hence, $U \circ F^{\text{op}} : \mathsf{Haus} \to \mathsf{Set}$ (where $U : \mathsf{Top} \to \mathsf{Set}$ is the forgetful functor) does not preserve monomorphisms. Hence, $G = U \circ F^{\text{op}}$ is not representable.

□

Exercise 2.1.v. The functor of Example 2.1.5(xi) that sends a category to its collection of isomorphisms is a subfunctor of the functor of Example 2.1.5(x) that sends a category to its collection of morphisms. Define a functor between the representing categories $\mathbb{I}$ and $\mathbb{2}$ that induces the corresponding monic natural transformations between these representable functors.

Example 2.1.5(xi) defined iso : $\mathsf{Cat} \to \mathsf{Set}$ as the functor that sends a small category to its set of isomorphisms, and Example 2.1.5(x) defined mor : $\mathsf{Cat} \to \mathsf{Set}$ as sending a small category to its set of morphisms.

We have that iso$\mathsf{C} \cong \text{Hom}(\mathbb{I}, \mathsf{C})$ and Mor$\mathsf{C} \cong \text{Hom}(\mathbb{2}, \mathsf{C})$ (as before). Hence, iso$\mathsf{C} \hookrightarrow$ Mor$\mathsf{C}$, i.e., $\text{Hom}(\mathbb{I}, \mathsf{C}) \hookrightarrow \text{Hom}(\mathbb{2}, \mathsf{C})$ is induced by the functor $F : \mathbb{2} \to \mathbb{I}$ given by mapping $0 \mapsto 0, 1 \mapsto 1$, and the morphism $! : 0 \to 1$ is mapped to the same morphism $0 \to 1 \in \mathbb{I}$:



Section 2.2. The Yoneda Lemma

Exercise 2.2.ii. Explain why the Yoneda lemma does not dualize to classify natural transformations from an arbitrary set-valued functor to a represented functor.

$\text{Hom}(F, \text{Hom}(c, -)) \cong Fc$ does not hold in general.

Consider $\mathsf{C} = \mathbb{1} = \{*, * \to *\}$ and let $c \in \mathsf{C}$ be the single object. Suppose $Fc = \varnothing$ for some set-valued functor $F : \mathsf{C} \to \mathsf{Set}$. Then,

$$\text{Hom}(\varnothing, \{*\}) = \{*\} \ncong \varnothing$$

Hence, the Yoneda lemma does not classify natural transformations from an arbitrary set-valued functor to a represented functor.

□

Exercise 2.2.iv. Prove the following strengthening of Lemma 1.2.3, demonstrating the equivalence between an isomorphism in a category and a **representable isomorphism** between the corresponding co- or contravariant represented functors: the following are equivalent:

(i) $f : x \to y$ is an isomorphism in C.

(ii) $f_* : \mathsf{C}(-, x) \Rightarrow \mathsf{C}(-, y)$ is a natural isomorphism.

(iii) $f^* : \mathsf{C}(y, -) \Rightarrow \mathsf{C}(x, -)$ is a natural isomorphism.

(i) $\Rightarrow$ (ii):

Supppose $f : x \to y \in \mathsf{C}$ is an isomorphism. By Example 1.4.7, $f_* : \mathsf{C}(-, x) \Rightarrow \mathsf{C}(-, y)$ is a natural transformation. For each $c \in \mathsf{C}$, define $(f_*)_c^{-1} := ((f_*)_c)^{-1}$. This exists since f is an isomorphism and hence each component is an isomorphism so f_* is a natural isomorphism.

(ii) $\Rightarrow$ (iii):

Suppose $f_* : \mathsf{C}(-, x) \Rightarrow \mathsf{C}(-, y)$ is a natural isomorphism. By commutativity of figure 1.4.8, then $- \cdot f = f^*$ must also be an isomorphism, so f^* is a natural isomorphism.

(iii) $\Rightarrow$ (i):

Suppose $f^* : \mathsf{C}(y, -) \Rightarrow \mathsf{C}(x, -)$ is a natural isomorphism. By the Yoneda embedding (*Corollary 2.2.8*), the functor $y' : \mathsf{C}^{\text{op}} \hookrightarrow \mathsf{Set}^{\mathsf{C}}$ is fully faithful. Then, y' defines a local bijection between hom-sets given by

$$\mathsf{C}(x, y) \xrightarrow{\cong} \text{Hom}(\mathsf{C}(y, -), \mathsf{C}(x, -))$$

Also, fully faithful functors reflect isomorphisms. Since $f^* : \mathsf{C}(y, -) \Rightarrow \mathsf{C}(x, -)$ is a natural isomorphism, this corresponds to an isomorphism $f : x \to y \in \mathsf{C}$. Thus, $f : x \to y$ is an isomorphism in C.

□

Exercise 2.2.v. By the Yoneda lemma, natural endomorphisms of the contravariant power set functor $P : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ correspond bijectively to endomorphisms of its representing object $\Omega = \{\bot, \top\}$. Describe the natural endomorphisms of P that correspond to each of the four elements of $\mathrm{Hom}(\Omega, \Omega)$. Do these functions induce natural endomorphisms of the covariant power set functor?

The contravariant power set functor $P : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ is defined by sending a set $A \in \mathsf{Set}^{\mathrm{op}}$ to its power set PA and a function $f : A \to B$ to $Pf = f^{-1} : PB \to PA$. This is represented by the set $\Omega = \{\bot, \top\}$ with

$$\mathsf{Set}(A, \Omega) \cong PA \qquad (1)$$

defined by the bijection that associates a function $f : A \to \Omega$ with the subset $f^{-1}(\top)$ and associates any subset $A' \subseteq A$ to $\chi_{A'} : A \to \Omega$ that sends elements of A' to $\top$.

By the Yoneda lemma, since $P : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ is a functor and $\Omega \in \mathsf{Set}^{\mathrm{op}}$, then we have the natural isomorphism

$$\mathrm{Hom}(\mathsf{Set}^{\mathrm{op}}(-, \Omega), P) \cong P\Omega$$

By the isomorphism (1) above, then

$$\mathrm{Hom}(P, P) \cong \mathrm{Hom}(\Omega, \Omega)$$

The 4 elements of $\mathrm{Hom}(\Omega, \Omega)$ are as follows:

1. $\mathrm{id} : \top \mapsto \top, \bot \mapsto \bot;$
2. $\mathrm{true} : \top \mapsto \top, \bot \mapsto \top;$
3. $\mathrm{false} : \top \mapsto \bot, \bot \mapsto \bot;$
4. $\mathrm{swap} : \top \mapsto \bot, \bot \mapsto \top.$

By definition, $P(A) = \{S \mid S \subseteq A\}$ and let$f \in \mathrm{Hom}(A, \Omega) = \{\text{functions } A \to \Omega\}$. Then,

$$Ff = \{a \in A \mid f(a) = \top\}$$

$$GS(a) = \begin{cases} \top & a \in S \\ \bot & a \notin S \end{cases}$$

Now, $\mathrm{id} \in \mathrm{Hom}(\Omega, \Omega)$ clearly corresponds to the natural endomorphism $\mathrm{id} : P \Rightarrow P$. The map true corresponds to the natural endomorphism $\overline{\mathrm{true}} : P \Rightarrow P$ that is defined by the following components (for each $A \in P$):

$$\overline{\mathrm{true}}_A : P(A) \to P(A)$$
$$S \mapsto A$$

i.e., any subset of A is now "true" and all elements are also in A.

Analogously, the map false corresponds to the natural endomorphism $\overline{\mathrm{false}} : P \Rightarrow P$ that is defined by the following components (for each $A \in P$):

$$\overline{\mathrm{false}}_A : P(A) \to P(A)$$
$$S \mapsto \varnothing$$

i.e., any subset of A is now "false" the new set contains no elements.

Lastly, the map swap corresponds to the natural endomorphism $\overline{\mathrm{swap}} : P \Rightarrow P$ that is defined by the following components (for each $A \in P$):

$$\overline{\mathrm{swap}}_A : P(A) \to P(A)$$
$$S \mapsto S^C$$

i.e., if the subset were "true", it is now "false", and vice versa.

□

Section 2.3. Universal Properties and Universal Elements

Exercise 2.3.iii. The set B^A of functions from a set A to a set B represents the contravariant functor $\mathsf{Set}(- \times A, B) : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$. The universal element for this representation is a function

$$\mathrm{ev} : B^A \times A \to B$$

called the evaluation map. Define the evaluation map and describe its universal property, in analogy with the universal bilinear map $\otimes$ of Example 2.3.7.

Define the evaluation map $\mathrm{ev} : B^A \times A \to B$ by the following.

$$\mathrm{ev} : (f, a) \mapsto f(a)$$

for all $f \in B^A, a \in A$.

As the contravariant functor $\mathsf{Set}(- \times A, B)$ is represented by B^A, then for all $D \in \mathsf{Set}$,

$$\mathsf{Set}(D, B^A) \cong \mathsf{Set}(D \times A, B)$$

This natural isomorphism is defined by associating any function $g : D \times A \to B$, with the function $\bar{g} : D \to B^A$ such that

$$\bar{g}(d) : A \to B$$

defined by

$$\bar{g}(d)(a) = g(d, a)$$

And on the other hand, it associates a function $h : D \to B^A$ with

$$\hat{h} : D \times A \to B$$

defined by

$$\hat{h}(d, a) = h(d)(a)$$

Let $f : D \times A \to B$. The natural isomorphism associates f with a function $\bar{f} : D \to B^A$. Consider the naturality square induced by $\bar{f}$ that must hold:

$$\begin{array}{ccc} \mathsf{Set}(B^A, B^A) & \xrightarrow{\cong} & \mathsf{Set}(B^A \times A, B) \\ {\scriptstyle \bar{f}^*} \downarrow & & \downarrow {\scriptstyle (\bar{f} \times \mathrm{id}_A)^*} \\ \mathsf{Set}(D, B^A) & \xrightarrow[\cong]{} & \mathsf{Set}(D \times A, B) \end{array}$$

where $\bar{f}^*$ is the usual notation for pre-composition by $\bar{f}$.

Now, tracing 1_{B^A} reveals that f factors uniquely through ev along a unique $\bar{f} : D \to B^A$ such that

$$\begin{array}{ccc} D \times A & \xrightarrow{f} & B \\ & {\scriptstyle \bar{f} \times \mathrm{id}_A} \searrow & \uparrow {\scriptstyle \mathrm{ev}} \\ & & B^A \times A \end{array}$$

commutes.

Universal property of ev:

For arbitrary $D \in \mathsf{Set}$ and function $f : D \times A \to B$, there exists a unique $\bar{f} : D \to B^A$ such that

$$f = \mathrm{ev} \circ (\bar{f} \times \mathrm{id}_A)$$

□

Section 2.4. The Category of Elements

Exercise 2.4.ii. Characterize the terminal objects of C/c.

C/c is the slice category over the object $c \in \mathsf{C}$, formally denoted as $\int \mathsf{C}(-, c)$. Objects are morphisms $f : x \to c$. Morphisms in C/c from $f : x \to c$ to $g : y \to c$ is a morphism $h : x \to y$ such that

$$gh = f$$

Let $f \in \mathsf{C}/c$ be arbitrary. By definition, there exists a morphism $g : f \to \mathrm{id}_c$ if and only if

$$f = \mathrm{id}_c g = g$$

Hence, for every $f \in \mathsf{C}/c$, there exists a unique morphism (f itself) such that $f \to \mathrm{id}_c \in \mathsf{C}/c$. So we have:

$$\begin{array}{ccc} x & \xrightarrow{f} & c \\ & \searrow^{f} & \downarrow{\scriptstyle \mathrm{id}_c} \\ & & c \end{array}$$

and id_c is the terminal object of C/c.

□

Exercise 2.4.iv. Explain the sense in which the Sierpinski space is the universal topological space with an open subset.

The Sierpinski space is a topological space with two points, one open and one closed. Let the underlying set of S be denoted $\{0, 1\}$ and its open sets $\{\varnothing, \{1\}, \{0, 1\}\}$. By Section 2.1, there exists a bijection

$$\mathsf{Top}(X, S) \cong \mathcal{O}(X)$$

where X is a topological space and $\mathcal{O} : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$ is the functor that sends a space to its set of open subsets. This bijection is defined by associating a function $f : X \to S \in \mathsf{Top}^{\mathrm{op}}(X, S)$ with the open subset $f^{-1}(\{1\}) \in \mathcal{O}(X)$.

The objects in the category of elements $\int \mathcal{O}$ of the contravariant functor $\mathcal{O}$ are pairs (S, X) where $S \in \mathsf{Top}, X \in \mathcal{O}(S)$. The terminal object is the Sierpinski space with open set $\{1\} \subset S$. This is because for any open set $A \subseteq X$, there exists a *unique* continuous function

$$\begin{aligned} \chi_A :& X \to S \\ \text{s.t. } & A = \chi_A^{-1}(\{1\}) \end{aligned}$$

Hence, $(S, \{1\}) \in \int \mathcal{O}$ is the terminal object, so the Sierpinski space is the universal topological space with an open subset.

□

Exercise 2.4.vi. For a locally small category C, regard the two-sided represented functor $\mathrm{Hom}(-,-) : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$ as a covariant functor of its domain $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$. The category of elements of Hom is called the twisted arrow category. Justify this name by describing its objects and morphisms.

The category of elements of Hom, denoted $\int \mathrm{Hom}$ consists of the following:

Objects: Pairs (c, x) for $c = (c_1, c_2) \in \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$, $x \in \mathrm{Hom}(c_1, c_2)$. That is, objects are morphisms $f \in \mathsf{C}$.

Morphisms:

By definition, if $h : (c, f) \to (c', g)$ is a morphism in $\int \mathrm{Hom}$, for $c = (x, y) \in \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$, $c' = (w, z) \in \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$, then $Fh(f) = g$ where F is the functor $\mathrm{Hom}(-,-)$.

Denote $h_1 : w \to x \in \mathsf{C}^{\mathrm{op}}$ and $h_2 : y \to z \in \mathsf{C}$. If $Fh(f) = g$, this implies that

$$(h_1^*)(f h_{2_*}) = g$$

Hence,

$$h_2 f h_1 = g$$

That is, this commutes:

$$\begin{array}{ccc} x & \xleftarrow{h_1} & w \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ y & \xrightarrow[h_2]{} & z \end{array}$$

In summary: for any $f, g \in \int \mathrm{Hom}$, morphisms between f and g are pairs (j, k) such that this commutes:

$$\begin{array}{ccc} x & \xleftarrow{j} & w \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ y & \xrightarrow[k]{} & z \end{array}$$

The arrows of j and k are reversed or "twisted", justifying the name of the category as "twisted arrow category": it is a category of arrows in which morphisms between them twist the arrows.

$\square$