1 Chapter 1: Categories, functors, natural transformations

Section 1.2. Duality

Exercise 1.2.ii. (i) Show that a morphism $f : x \to y$ is a split epimorphism in a category C if and only if for all $c \in \mathsf{C}$, post-composition $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ defines a surjective function.

$\Rightarrow$: Suppose $f : x \to y$ is a split epimorphism. That is, there exists some $h : y \to x$ such that

$$fh = 1_y$$

Let $c \in \mathsf{C}$ and $g \in \mathsf{C}(c, y)$ be arbitrary. Then, $hg : c \to x$ hence $hg \in \mathsf{C}(c, x)$. By definition of f_*, then

$$f_*(hg) = fhg = 1_y g = g$$

Since $g \in \mathsf{C}(c, y)$ was arbitrary, it follows that f_* is surjective for all $c \in \mathsf{C}$.

$\Leftarrow$: Conversely, suppose that $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ is surjective for every $c \in \mathsf{C}$.
Take $c = y$. Then, since $f_* : \mathsf{C}(y, x) \to \mathsf{C}(y, y)$ is surjective, then there exists some $h \in \mathsf{C}(y, x)$ such that

$$f_* h = 1_y \quad \rightsquigarrow \quad fh = 1_y$$

so f is a split epimorphism.

$\square$

(ii) Argue by duality that f is a split monomorphism if and only if for all $c \in \mathsf{C}$, pre-composition $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ is a surjective function.

Since (i) holds for any category C, then the statement holds in the category C^{op}. That is, $f^{\mathrm{op}} : y \to x$ is a split epimorphism in C^{op} if and only if $f_*^{\mathrm{op}} : \mathsf{C}^{\mathrm{op}}(c, y) \to \mathsf{C}^{\mathrm{op}}(c, x)$ is surjective for all $c \in \mathsf{C}^{\mathrm{op}}$. This translates to $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ since $\mathsf{C}^{\mathrm{op}}(c, x) = \mathsf{C}(x, c)$. That is, post-composition with f^{op} in C^{op} is the same as pre-composition with f in its opposite category C.
Also, $f^{\mathrm{op}} : y \to x$ is a split epimorphism iff there exists some $h^{\mathrm{op}} : x \to y$ such that

$$x \xrightarrow{h^{\mathrm{op}}} y \xrightarrow{f^{\mathrm{op}}} x$$

where $f^{\mathrm{op}} h^{\mathrm{op}} = 1_x$. This implies that

$$x \xleftarrow{h} y \xleftarrow{f} x$$

and $hf = 1_x$. By definition, this means that $f : x \to y$ is a split monomorphism.
Hence, by (i), f is a split monomorphism if and only if $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ is surjective.

$\square$

Exercise 1.2.v. Show that the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both a monomorphism and an epimorphism in the category Ring of rings. Conclude that a map that is both monic and epic need not be an isomorphism.

Let $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$ denote the inclusion.

- ι is a monomorphism:
  Let R be an arbitrary unital ring in the category Ring. Let $h, k : R \rightrightarrows \mathbb{Z}$ be ring homomorphisms such that $\iota h = \iota k$. Let $r \in R$ be arbitrary. Then,

  $$\iota h(r) = \iota k(r) \quad \rightsquigarrow \quad h(r) = k(r)$$

  since ι is the inclusion map. Hence, $h = k$. Thus, ι is a monomorphism.

- ι is an epimorphism:
  Let S be an arbitrary unital ring and $h, k : \mathbb{Q} \rightrightarrows S$ be such that $h\iota = k\iota$.
  Claim: If $h, k : \mathbb{Q} \rightrightarrows S$ are ring homomorphisms such that $h(n) = k(n)$ for all $n \in \mathbb{Z}$, then $h = k$. i.e., ring homomorphisms from $\mathbb{Q}$ are uniquely determined by the image of $\mathbb{Z}$.
  Proof of claim: Let $q = \frac{a}{b} \in \mathbb{Q}$ be arbitrary. Then,

$$\begin{aligned} h\left(\frac{a}{b}\right) &= h(a)h\left(\frac{1}{b}\right) \\ &= k(a)h\left(\frac{1}{b}\right) \\ &= k\left(\frac{a}{b}b\right)h\left(\frac{1}{b}\right) \\ &= k\left(\frac{a}{b}\right)k(b)h\left(\frac{1}{b}\right) = k\left(\frac{a}{b}\right)h(b)h\left(\frac{1}{b}\right) \\ &= k\left(\frac{a}{b}\right)h(1) \\ \rightsquigarrow \quad h\left(\frac{a}{b}\right) &= k\left(\frac{a}{b}\right) \end{aligned}$$

  As $q \in \mathbb{Q}$ was arbitrary, then $h = k$.
  Now, since $h\iota = k\iota$ by assumption, then $h(n) = k(n)$ for all $n \in \mathbb{Z}$. By the claim, then $h = k$. Thus, ι is an epimorphism.

Therefore, even though $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic and monic, it is not an isomorphism (it is not surjective).

□

Exercise 1.2.vi. Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism. Argue by duality that a split monomorphism that is an epimorphism is also an isomorphism.

Let $f : x \to y$ be an arbitrary morphism in a category C such that:

- f is a monomorphism: for all $h, k : w \rightrightarrows x$, if $fh = fk$, then $h = k$.
- f is a split epimorphism: there exists some $g : y \to x$ such that $fg = 1_y$.

Now, consider $fgf : x \to y$. Since $fg = 1_y$, then

$$fgf = 1_y f = f 1_x$$

That is, $f(gf) = f(1_x)$. Since f is a monomorphism and $gf, 1_x : x \rightrightarrows x$, then $gf = 1_x$.
Therefore, there exists some $g : y \to x$ such that $fg = 1_y$ and $gf = 1_x$, so f is an isomorphism.

Now, since the above statement holds for all categories C, then it also holds in C^{op}. Let $f^{\mathrm{op}} : y \to x$ be a morphism in C^{op} such that f^{op} is a monomorphism and a split epimorphism. Then, by above, f^{op} is an isomorphism which implies that f is also an isomorphism (notion of isomorphism is self-dual). Now, f^{op} is a monomorphism $\rightsquigarrow$ f is an epimorphism and f^{op} is a split epimorphism $\rightsquigarrow$ f is a split monomorphism as follows.

- f^{op} is a monomorphism: for all $h^{\mathrm{op}}, k^{\mathrm{op}} : w \rightrightarrows y$, if $f^{\mathrm{op}}h^{\mathrm{op}} = f^{\mathrm{op}}k^{\mathrm{op}}$, then $h^{\mathrm{op}} = k^{\mathrm{op}}$.
  – This is precisely the statement that $f : x \to y$ is such that for all $h, k : y \rightrightarrows w$, if $hf = kf$, then $h = k$. That is, f is an epimorphism.
- f^{op} is a split epimorphism: there exists some $g^{\mathrm{op}} : x \to y$ such that $f^{\mathrm{op}}g^{\mathrm{op}} = 1_x$.
  – As in exercise 1.2.ii (ii), then f is a split monomorphism.

Therefore, if f is an epimorphism and a split monomorphism, then f is an isomorphism.

□

Section 1.3. Functoriality

Exercise 1.3.iii. Find an example to show that the objects and morphisms in the image of a functor $F : \mathsf{C} \to \mathsf{D}$ do not necessarily define a subcategory of D.

Let C be the category defined by 4 objects x, y, z, w and morphisms $x \to y$, $z \to w$ and the identity morphisms.

Let D be the category defined by 3 objects a, b, c and morphisms $a \to b$, $b \to c$, $a \to c$, and the identity morphisms.

Consider the functor $F : \mathsf{C} \to \mathsf{D}$ given by the following

$$F(x) = a, F(y) = b = F(z), \text{ and } F(w) = c$$

and F extends uniquely to the morphisms. The image of F is not a category. It consists of objects a, b, c and morphisms $Fx \to Fy : a \to b$, $Fz \to Fw : b \to c$ and the identity morphisms. But, for $F(\mathsf{C})$ to be a category, there would also need to be a composite morphism $a \to c$. By this construction, such a morphism does not exist. Hence, the image of F does not define a subcategory of D.

□

Exercise 1.3.viii. Lemma 1.3.8. shows that functors preserve isomorphisms. Find an example to demonstrate that functors need not **reflect isomorphisms**: that is, find a functor $F : \mathsf{C} \to \mathsf{D}$ and a morphism f in C so that Ff is an isomorphism in D but f is not an isomorphism in C.

Let $\mathsf{C} = \mathsf{Top}_*$ (category of based topological spaces) and $\mathsf{D} = \mathsf{Group}$. Let $F : \mathsf{C} \to \mathsf{D}$ be the functor $F = \pi_1$. Consider the morphism $\iota : S^1 \hookrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$ in Top_* where both spaces have basepoint $(1,0)$. Then, clearly $S^1 \ncong \mathbb{R}^2 \setminus \{(0,0)\}$ in Top_*. However, $\pi_1(S^1) \cong \pi_1(\mathbb{R}^2 \setminus \{(0,0)\} \cong \mathbb{Z}$. Hence, $F\iota$ is an isomorphism in Group but $\iota \in \mathsf{Top}_*$ is not an isomorphism. Therefore, functors need not reflect isomorphisms.

□

Exercise 1.3.ix. For any group G, we may define other groups:

- the center $Z(G) = \{h \in G \mid hg = gh \ \forall g \in G\}$, a sbgroup of G,
- the commutator subgroup $C(G)$, the subgroup generated by elements $ghg^{-1}h^{-1}$ for any $g, h \in G$, and
- the automorphism group $\mathrm{Aut}(G)$, the group of isomorphisms $\varphi : G \to G$ in Group.

Trivially, all three constructions define a functor from the discrete category of groups (with only identity morphisms) to Group. Are these constructions functorial in

- the isomorphisms of groups? That is, do they extend to functors $\mathsf{Group}_{\text{iso}} \to \mathsf{Group}$?
- the epimorphisms of groups? That is, do they extend to functors $\mathsf{Group}_{\text{epi}} \to \mathsf{Group}$?
- all homomorphisms of groups? That is, do they extend to functors $\mathsf{Group} \to \mathsf{Group}$?

First, note that the constructions send objects (groups) to objects. Hence, to check functoriality, it suffices to check the morphism axioms.

(1) $\mathsf{Group}_{\text{iso}} \to \mathsf{Group}$:
In $\mathsf{Group}_{\text{iso}}$, if $f : G \to H$ is a morphism, then $G \cong H$ (G and H are isomorphic groups). By results in group theory, then

$$Z(G) \cong Z(H)$$
$$C(G) \cong C(H)$$
$$\mathrm{Aut}(G) \cong \mathrm{Aut}(H)$$

In particular, denoting F_1 as the functor $Z(-)$, F_2 as the functor $C(-)$, and F_3 as the functor $\mathrm{Aut}(-)$, then in Group, there are morphisms $F_1 f : Z(G) \to Z(H), F_2 f : C(G) \to C(H), F_3 f : \mathrm{Aut}(G) \to \mathrm{Aut}(H)$ (in fact, they are isomorphisms).

To check the functoriality axioms, let $f, g \in \mathsf{Group}_{\mathrm{iso}}$ be such that $f : G \to H, g : H \to J$ for groups G, H, J and f, g group isomorphisms. Then, $F_1 g F_1 f = F_1(gf)$ by construction and the same holds for F_2, F_3. Also, for any $G \in \mathsf{Group}_{\mathrm{iso}}$, $F_i(1_G) = 1_{F_{i_G}}$. Hence, these constructions are functorial in the isomorphisms of groups.

(2) $\mathsf{Group}_{\mathrm{epi}} \to \mathsf{Group}$:

Suppose $f : G \twoheadrightarrow H \in \mathsf{Group}_{\mathrm{epi}}$ is a surjective group homomorphism.

$Z(-)$:

Define the action of $F_1 = Z(-)$ on homomorphism f by $f|_{Z(G)} : Z(G) \to Z(H)$. This is indeed a homomorphism. Let $g \in Z(G)$ be arbitrary. To see that $f(g) \in Z(H)$, let $h \in H$ be arbitrary. Then, since f is surjective, there exists some $g' \in G$ such that

$$f(g') = h$$

Then,

$$f(g)h = f(g)f(g')$$

Since f is a homomorphism, then

$$f(g)h = f(gg')$$

Since $g \in Z(G)$, then $gg' = g'g$ and hence

$$\begin{aligned} & f(g)h = f(g'g) \\ & f(g)h = f(g')f(g) \\ \Rightarrow \quad & f(g)h = hf(g) \end{aligned}$$

Since $h \in H$ was arbitrary, then $f(g) \in Z(H)$ and $f : Z(G) \to Z(H)$ defines a homomorphism, so $(Z-) : \mathsf{Group}_{\mathrm{epi}} \to \mathsf{Group}$ is a functor.

$\mathrm{Aut}(-)$:

If $f : G \to H$ is a homomorphism, what property must it have for $\mathrm{Aut}(f)$ to be defined? Let $\varphi \in G$. If $\mathrm{Aut}(f) : \mathrm{Aut}(G) \to \mathrm{Aut}(H)$ exists, then the following must hold.

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G \\ {\scriptstyle f}\downarrow \uparrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f} \\ H & \dashrightarrow & H \end{array}$$

That is, f^{-1} must exist, so f must be an isomorphism. For every $\varphi \in \mathrm{Aut}(G)$, define

$$\mathrm{Aut}(f)(\varphi) = f \circ \varphi \circ f^{-1}$$

This is a functor if and only if f is an isomorphism. Suppose $G \xrightarrow{f} H \xrightarrow{g} K$ are group isomorphisms. Then, for any $\varphi \in \mathrm{Aut}(G)$, the following holds.

$$\begin{aligned} \mathrm{Aut}(g \circ f)(\varphi) &= (g \circ f) \circ \varphi \circ (g \circ f)^{-1} \\ &= g \circ f \circ \varphi \circ f^{-1} \circ g^{-1} \\ &= \mathrm{Aut}(g)(f \circ \varphi \circ f^{-1}) \\ \Rightarrow \quad \mathrm{Aut}(g \circ f)(\varphi) &= \mathrm{Aut}(g) \circ \mathrm{Aut}(f)(\varphi) \end{aligned}$$

Also, it is clear that $\mathrm{Aut}(\mathrm{id}_G) = \mathrm{id}_{\mathrm{Aut}(G)}$. Hence, $\mathrm{Aut}(-)$ defines a functor if and only if f is an isomorphism. That is, $\mathrm{Aut}(-) : \mathsf{Group}_{\mathrm{epi}} \to \mathsf{Group}$ does not define a functor.

(3) $\mathsf{Group} \to \mathsf{Group}$:

Suppose $f : G \to H \in \mathsf{Group}$ is a group homomorphism. From above, since f is not necessarily an isomorphism, then $\mathrm{Aut}(-) : \mathsf{Group} \to \mathsf{Group}$ is not a functor. Also, $Z(-) : \mathsf{Group} \to \mathsf{Group}$ does not define a functor. To see this, consider $\mathbb{Z}/2\mathbb{Z} = S_2 \hookrightarrow S_3$ given by

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Since $Z(S_2) = S_2$, this does not preserve centers (as $(213) \notin Z(S_3)$), so $Z(-) : \mathsf{Group} \to \mathsf{Group}$ is not a functor.

$\square$

Section 1.4. Naturality

Exercise 1.4.i. Suppose $\alpha : F \Rightarrow G$ is a natural isomorphism. Show that the inverses of the component morphisms define the components of a natural isomorphism $\alpha^{-1} : G \Rightarrow F$.

Let C, D be categories and $F, G : \mathsf{C} \rightrightarrows \mathsf{D}$ be functors such that $\alpha : F \Rightarrow G$ is a natural isomorphism. By definition, then $\alpha_c : Fc \to Gc$ is an isomorphism for every $c \in \mathsf{C}$. That is, for each $c \in \mathsf{C}$, there exists a morphism $\alpha_c^{-1} : Gc \to Fc \in \mathsf{C}$ such that

$$\alpha_c \alpha_c^{-1} = 1_{Gc}$$

and

$$\alpha_c^{-1} \alpha_c = 1_{Fc}$$

Define $\alpha^-1 : G \Rightarrow F$ using the maps α_c^{-1} as the components of α. Now, let $f : c \to c'$ be an arbitrary morphism in C. Since $\alpha : F \Rightarrow G$, then the following commutes:

$$\begin{array}{ccc} Fc & \xrightarrow{\alpha_c} & Gc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fc' & \xrightarrow[\alpha_{c'}]{} & Gc' \end{array}$$

That is, for all $c, c' \in \mathsf{C}$,

$$Gf\alpha_c = \alpha_{c'} Ff$$

Pre-composing both sides by α_c^{-1}, we obtain

$$\begin{aligned} Gf\alpha_c\alpha_c^{-1} &= \alpha_{c'} Ff \alpha_c^{-1} \\ Gf 1_{Gc} &= \alpha_{c'} Ff \alpha_c^{-1} \end{aligned}$$

Post-composing by $\alpha_{c'}^{-1}$, then

$$\begin{aligned} & \alpha_{c'}^{-1} Gf = \alpha_{c'}^{-1}\alpha_{c'} Ff \alpha_c^{-1} \\ \Rightarrow \quad & \alpha_{c'}^{-1} Gf = Ff \alpha_c^{-1} \end{aligned}$$

That is, the following commutes for all $f : c \to c'$.

$$\begin{array}{ccc} Fc & \xleftarrow{\alpha_c^{-1}} & Gc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fc' & \xleftarrow[\alpha_{c'}^{-1}]{} & Gc' \end{array}$$

As $c \in \mathsf{C}$ was arbitrary, then the components α_c^{-1} define a natural transformation $\alpha^{-1} : G \Rightarrow F$. Also, since each α_c^{-1} is an isomorphism, then α^{-1} is in fact a natural isomorphism.

□

Exercise 1.4.iv. In the notation of Example 1.4.7, prove that distinct parallel morphisms $f, g : c \rightrightarrows d$ define distinct natural transformations

$$f_*, g_* : \mathsf{C}(-, c) \Rightarrow \mathsf{C}(-d,) \text{ and } f^*, g^* : \mathsf{C}(d, -) \Rightarrow \mathsf{C}(c, -)$$

by post- and pre-composition.

By Example 1.4.7 (p. 27), f_*, g_*, f^*, g^* define natural transformations. It remains to show that distinct $f, g : c \rightrightarrows d$ define distinct natural transformations in both cases.

Suppose $f_* = g_*$ are such that $\mathsf{C}(-,c) \underset{g_*}{\overset{f_*}{\rightrightarrows}} \mathsf{C}(-,d)$. Since $f_* = g_*$, then $(f_*)_c = (g_*)_c$. Hence, applying this to the morphism $\mathrm{id}_c \in \mathsf{C}(c,c)$,

$$\begin{aligned} (f_*)_c(\mathrm{id}_c) &= (g_*)_c(\mathrm{id}_c) \\ \Rightarrow \quad f &= g \end{aligned}$$

Thus, if $f \neq g$, then f_* and g_* define distinct natural transformations.

Analogously, suppose that $f^* = g^*$ are such that $\mathsf{C}(c,-) \underset{g^*}{\overset{f^*}{\rightrightarrows}} \mathsf{C}(d,-)$. Since $f^* = g^*$, then $(f^*)_c = (g^*)_c$. Hence, applying this to the morphism $\mathrm{id}_c \in \mathsf{C}(c,c)$, in the same way as above, $f = g$.

□

Section 1.5. Equivalence of categories

Exercise 1.5.xi. Consider the functors $\mathsf{Ab} \to \mathsf{Group}$ (inclusion), $\mathsf{Ring} \to \mathsf{Ab}$ (forgetting the multiplication), $(-)^\times$: $\mathsf{Ring} \to \mathsf{Group}$ (taking the group of units), $\mathsf{Ring} \to \mathsf{Rng}$ (inclusion), $\mathsf{Field} \to \mathsf{Ring}$ (inclusion), and $\mathsf{Mod}_R \to \mathsf{Ab}$ (forgetful). Determine which functors are full, which are faithful, and which are essentially surjective. Do any define an equivalence of categories?

A summary:

Functor	Full	Faithful	Essentially surjective	Equivalence
$\mathsf{Ab} \hookrightarrow \mathsf{Group}$	✓Yes	✓Yes	× No	× No
$\mathsf{Ring} \to \mathsf{Ab}$	× No	✓Yes	× No	× No
$(-)^\times : \mathsf{Ring} \to \mathsf{Group}$	× No	× No	× No *	× No
$\mathsf{Ring} \hookrightarrow \mathsf{Rng}$	× No	✓Yes	× No	× No
$\mathsf{Field} \hookrightarrow \mathsf{Ring}$	✓Yes	✓Yes	× No	× No
$\mathsf{Mod}_R \to \mathsf{Ab}$	× No	✓Yes	✓Yes	× No

First, note that by *Theorem 1.5.9* (characterization of category equivalences), a functor defines an equivalence of categories if and only if it is fully faithful and essentially surjective.

1. $\mathsf{Ab} \hookrightarrow \mathsf{Group}$.
   - $\mathsf{Ab} \hookrightarrow \mathsf{Group}$ is fully faithful.
     The inclusion map is clearly injective on objects. Also, any morphism in Ab is also a morphism in Group. Hence, Ab is a full subcategory of Group, and hence the inclusion functor is full and faithful.

   - $\mathsf{Ab} \hookrightarrow \mathsf{Group}$ is not essentially surjective because there exist non-abelian groups (e.g., the dihedral group D_6).

   $\rightsquigarrow$ since $\mathsf{Ab} \hookrightarrow \mathsf{Group}$ is not essentially surjective, it is not an equivalence of categories.

2. $\mathsf{Ring} \to \mathsf{Ab}$.
   Denote $F : \mathsf{Ring} \to \mathsf{Ab}$.
   - F is not full.
     There does not exist a ring homomorphism $\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$ but there does exist a group homomorphism $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$ given by the trivial homomorphism, i.e., $[x] \mapsto 0$ for all $x \in \mathbb{Z}/n\mathbb{Z}$. Hence, F is not surjective on hom-sets, so F is not full.

   - F is faithful.
     Let $\varphi : R \to S$ be an arbitrary ring homomorphism in Ring. Then, φ is a group homomorphism of abelian groups with additional structure. Thus, if $\varphi \neq \psi \in \mathsf{Ring}$, then $F\varphi \neq F\psi$ in Ab. That is $\varphi \neq \psi$ in Ab. Hence,

     $$\mathsf{Ring}(x, y) \to \mathsf{Ab}(x, y) \text{ is injective}$$

     so F is a faithful functor.

   - F is not essentially surjective.
     There exist abelian groups that cannot admit a ring structure (with unity). First note that for any finite abelian group, it is the product of finite cyclic groups, and hence it can admit a ring structure. Thus, the only possible candidates for groups that cannot admit a ring structure are infinite groups.

Consider $G = \mathbb{Q}/\mathbb{Z} \in \mathsf{Ab}$. Any $r + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ has finite additive order. If $r = \frac{m}{n}$ for $m, n \in \mathbb{Z}, n > 0$, then

$$n(r + \mathbb{Z}) = m + \mathbb{Z} = \mathbb{Z} = e \quad \rightsquigarrow \quad \text{identity in } \mathbb{Q}/\mathbb{Z}$$

Thus, $|r + \mathbb{Z}|$ divides n and so $|r + \mathbb{Z}|$ is finite.
Also, notice that for any $n \in \mathbb{Z}$, $\frac{1}{n} \in \mathbb{Q}/\mathbb{Z}$ has additive order n.

Claim: $\mathbb{Q}/\mathbb{Z}$ cannot admit a ring structure.
To see this, for the purpose of contradiction suppose that $\mathbb{Q}/\mathbb{Z}$ is a ring with unity. Since every element of $\mathbb{Q}/\mathbb{Z}$ has finite additive order, then as a ring, $\mathbb{Q}/\mathbb{Z}$ must have finite characteristic, say char $\mathbb{Q}/\mathbb{Z} = b$ where b is the smallest positive number such that

$$\begin{aligned} 1 + 1 + \cdots + 1 = b \cdot 1 = 0 & \\ \Rightarrow \quad & b \cdot x = 0 \text{ for all } x \in \mathbb{Q}/\mathbb{Z} \end{aligned}$$

Hence, every $x \in \mathbb{Q}/\mathbb{Z}$ must have additive order that divides b.
This is a contradiction to the fact that for any $n \in \mathbb{Z}$, $|\frac{1}{n}| = n$. That is, the additive orders of elements of $\mathbb{Q}/\mathbb{Z}$ are unbounded, and hence cannot be bounded by char $\mathbb{Q}/\mathbb{Z}$. Therefore, $\mathbb{Q}/\mathbb{Z}$ cannot admit a ring structure.
Therefore, there exist abelian groups that cannot admit a ring structure, so the functor $F : \mathsf{Ring} \to \mathsf{Ab}$ is not essentially surjective on objects.

$\rightsquigarrow$ since F is not full/not essentially surjective, it is not an equivalence of categories.

3. $(-)^\times : \mathsf{Ring} \to \mathsf{Group}$.

- $(-)^\times$ is not full.
  In Ring, $\mathbb{Z}$ is the initial ring (Example 1.6.15). Hence, for any $R \in \mathsf{Ring}$, there exists a unique ring homomorphism
  $$\varphi : \mathbb{Z} \to R$$
  Let $R = \mathbb{Z}/p\mathbb{Z}$ for some prime $p > 2$. Then, there exists a unique ring homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. Applying $(-)^\times$, $\mathbb{Z}^\times \cong \mathbb{Z}/2\mathbb{Z}$ and $(\mathbb{Z}/p\mathbb{Z})^\times \cong \mathbb{Z}/(p-1)\mathbb{Z}$. But, there exist two distinct group homomorphisms
  $$\mathbb{Z}^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$$
  That is, there is the trivial group homomorphism and the group homomorphism that sends $1 \mapsto x$ where $x \in (\mathbb{Z}/p\mathbb{Z})^\times$ is the unique element of order 2. Therefore,
  $$(-)^\times : \mathsf{Ring}(x, y) \to \mathsf{Group}(x, y) \text{ is not surjective.}$$
  That is, $(-)^\times$ is not full.

- $(-)^\times$ is not faithful.
  Let L be a field and consider the polynomial ring $L[x]$. Denote the automorphisms that fix L by:
  $$M := \{\varphi \in \mathrm{Aut}(L[x]) \mid \varphi(\ell) = \ell \ \forall \ell \in L\} = \{\varphi(x) = ax + b \mid a \in L^\times, b \in L\}$$
  Since $L[x]^\times = L^\times$, then for any $\varphi \in M$, $\varphi \stackrel{(-)^\times}{\longmapsto} 1_{L^\times}$. But there are many distinct elements in M. Therefore,
  $$\mathsf{Ring}(L[x], L[x]) \to \mathsf{Group}(L^\times, L^\times) \text{ is not injective (in general).}$$
  Hence, $(-)^\times$ is not faithful.

- $(-)^\times$ is not essentially surjective.
  $(-)^\times$ is not essentially surjective because $\mathbb{Z}/5\mathbb{Z}$ is not isomorphic to the group of units of any unital ring in Ring.
  For the purpose of contradiction, suppose $R \in \mathsf{Ring}$ is a ring with unity such that $|R^\times| = 5$. If 1 denotes the unital element, it follows that $(-1) \in R^\times$ as well. If $1 \neq -1$, then it would follow that $|R^\times|$ is even. Since $|R^\times| = 5$ is odd, then it follows that $1 = -1$. That is, R contains a subfield isomorphic to $\mathbb{Z}/2\mathbb{Z}$.
  Claim: char $R = 2$.
  If char $R \neq 2$, then for any $u \in R^\times$, there exists $-u \in R^\times$ and if $u = -u$, then $|R^\times|$ is either infinite or even. Since $|R^\times| = 5$, it follows that char $R = 2$.
  Now, char $R = 2$ implies that R may contain subfields of characteristic 2. None of these subfields can contain transcendentals since $|R^\times|$ is finite. Also, none of these subfields can be larger than $\mathbb{Z}/2\mathbb{Z}$ since such a subfield would contain a $(2^n - 1)^{st}$ root of unity, and $2^n - 1 \mid 5 \Leftrightarrow n = 1$.
  Thus, char $R = 2$ and the only subfields of characteristic 2 are $\mathbb{Z}/2\mathbb{Z}$. Now, let $u \in R^\times$ be arbitrary. That is, $u^5 = 1$ and hence the subgroup generated by u has order 5:
  $$|\langle u \rangle| = 5$$
  and u has 5 units. Also,
  $$\langle u \rangle \cong \mathbb{Z}/2\mathbb{Z}[x]/\langle f(x) \rangle$$
  where $f(x) \mid x^5 - 1$. Since $x^5 - 1$ is the product of distinct irreducibles, then we have the following cases:
  i. $f(x) = x - 1 \quad \rightsquigarrow \quad \langle u \rangle \cong \mathbb{Z}/2\mathbb{Z}$ (1 unit)
  ii. $f(x) = \frac{x^5-1}{x-1} \quad \rightsquigarrow \quad \langle u \rangle \cong \mathbb{F}_{16}$ (15 units)
  iii. $f(x) = x^5 - 1 \quad \rightsquigarrow \quad \langle u \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{F}_{16}$ (15 units)
  In all cases, $|R^\times| = 5$ does not hold. This is a contradiction.
  Therefore, there does not exist a ring $R \in \mathsf{Ring}$ with group of units isomorphic to $\mathbb{Z}/5\mathbb{Z}$. Hence, $(-)^\times$ is not essentially surjective.

$\rightsquigarrow$ since $(-)^\times$ is not full/not faithful/not essentially surjective, it is not an equivalence of categories.

4. $\mathsf{Ring} \hookrightarrow \mathsf{Rng}$.
   - $\mathsf{Ring} \hookrightarrow \mathsf{Rng}$ is not full.
     In Ring, by Example 1.6.15, $\mathbb{Z}$ is the initial object. That is, for every $R \in \mathsf{Ring}$, there exists a unique ring homomorphism $\mathbb{Z} \to R$. On the other hand, in Rng, there exist 2 homomorphisms $\mathbb{Z} \to R$: the trivial (zero) homomorphism and the non-trivial homomorphism from Ring. Thus, $\mathsf{Ring} \hookrightarrow \mathsf{Rng}$ is not full (not surjective on hom-sets).
   - $\mathsf{Ring} \hookrightarrow \mathsf{Rng}$ is faithful.
     If $\varphi = \psi \in \mathsf{Ring}$, then clearly $\varphi = \psi$ as ring homomorphisms in Rng. Hence, $\mathsf{Ring} \hookrightarrow \mathsf{Rng}$ is injective on hom-sets, so it is faithful.
   - $\mathsf{Ring} \hookrightarrow \mathsf{Rng}$ is not essentially surjective.
     As an arbitrary ring without unity is not isomorphic to a ring with unity, then $\mathsf{Ring} \hookrightarrow \mathsf{Rng}$ is not essentially surjective.

   $\rightsquigarrow$ since $\mathsf{Ring} \hookrightarrow \mathsf{Rng}$ is not full/not essentially surjective, it is not an equivalence of categories.

5. $\mathsf{Field} \hookrightarrow \mathsf{Ring}$.
   - $\mathsf{Field} \hookrightarrow \mathsf{Ring}$ is fully faithful.
     The inclusion is injective on objects. Also, any field homomorphism is also a ring homomorphism, so Field is a full subcategory of Ring. Hence, $\mathsf{Field} \hookrightarrow \mathsf{Ring}$ is fully faithful.

- $\mathsf{Field} \hookrightarrow \mathsf{Ring}$ is not essentially surjective.
  There exist rings that are not isomorphic to any field.

$\rightsquigarrow$ since $\mathsf{Field} \hookrightarrow \mathsf{Ring}$ is not essentially surjective, it is not an equivalence of categories.

6. $\mathsf{Mod}_R \to \mathsf{Ab}$.
   Denote $G : \mathsf{Mod}_R \to \mathsf{Ab}$.

- G is not full.
  Let R be an arbitrary unital ring. Viewing R as an R-module, then an R-module homomorphism $\varphi : R \to R$ is uniquely determined by $\varphi(1)$. That is, $\text{End}_R R \cong R$. In general, group homomorphisms $R \to R$ are *only uniquely determined by the image of one element when the group is free.* Hence, G is not full.

- G is faithful.
  Let $\varphi : M \to N$ be an arbitrary R-module homomorphism in Mod_R. Then, φ is a group homomorphism of abelian groups with additional structure. Hence, if $\varphi \neq \psi \in \mathsf{Mod}_R$, then $G\varphi \neq G\psi$ in Ab. That is, $\varphi \neq \psi$ in Ab. Hence, G is injective on hom-sets, so G is faithful.

- G is essentially surjective.
  Let R be any unital ringand $H \in \mathsf{Ab}$ any abelian group. Define H_R by the R-action given by

$$r \cdot h = h \text{ for all } r \in R, h \in H$$

  Then, H_R is an R-module and clearly $G(H_R) = H$. Hence, G is essentially surjective.

$\rightsquigarrow$ since G is not full (in general), it is not an equivalence of categories.

$\square$

Section 1.6. The art of the diagram chase

Exercise 1.6.ii. Show that any two terminal objects in a category are connected by a unique isomorphism.

Let C be a category and let $t, t' \in \mathsf{C}$ be two terminal objects. Then, by definition, there exists a unique morphism $f : t \to t' \in \mathsf{C}$ and a unique morphism $g : t' \to t \in \mathsf{C}$. Now, consider $gf : t \to t$. Since t is a terminal object, there can only be 1 morphism $t \to t$. Since $1_t : t \to t$, then it follows that $1_t = gf$ by uniqueness. In the same way, since t' is a terminal object and $fg : t' \to t'$, then $1_{t'} = fg$. Since f and g were unique and

$$fg = 1_{t'} \text{ and}$$
$$gf = 1_t$$

it follows that there exists a unique isomorphism connecting t and t'. As t, t' were arbitrary terminal objects, this holds for any two terminal objects in C.

□

Exercise 1.6.iii. Show that any faithful functor reflects monomorphisms. That is, if $F : \mathsf{C} \to \mathsf{D}$ is faithful, prove that if Ff is a monomorphism in D, then f is a monomorphism in C. Argue by duality that faithful functors also reflect epimorphisms. Conclude that in any concrete category, any morphism that defines an injection of underlying sets is a monomorphism and any morphism that defines a surjection of underlying sets is an epimorphism.

Let C, D be arbitrary categories and $F : \mathsf{C} \to \mathsf{D}$ a faithful functor. Suppose $f : x \to y \in \mathsf{C}$ is such that $Ff : Fx \to Fy \in \mathsf{D}$ is a monomorphism. Let $h, k : w \rightrightarrows x \in \mathsf{C}$ be such that

$$fh = fk$$

By functoriality, then

$$F(fh) = F(fk)$$
$$F(f)F(h) = F(f)F(k)$$

As Ff is a monomorphism, then it follows that

$$Fh = Fk$$

Since F is faithful, it is injective on morphisms and hence it follows that $h = k$. As $h, k : w \rightrightarrows x$ such that $fh = fk$ were arbitrary, it follows that $f \in \mathsf{C}$ is a monomorphism.

Similarly, if $Ff : Fx \to Fy$ is an epimorphism, let $h, k : y \rightrightarrows z$ be such that $hf = kf$. Then, again by functoriality $F(h)F(f) = F(k)F(f)$ and by F being epic, $Fh = Fk$. By faithfulness, then $h = k$. Hence, F also reflects epimorphisms.

Lastly, let C be a concrete category. By *Definition 1.6.17*, there exists a faithful functor $U : \mathsf{C} \to \mathsf{Set}$ (the forgetful functor). By the above results, if a morphism defines an injective (monic) map of underlying sets, it follows that it is a monomorphism in C; and if a morphism defines a surjective (epic) map of underlying sets, then it is an epimorphism in C.

□

Exercise 1.6.iv. Find an example to show that a faithful functor need not preserve epimorphisms. Argue by duality, or by another counterexample, that a faithful functor need not preserve monomorphisms.

The full embedding $F : \mathsf{Haus} \hookrightarrow \mathsf{Top}$ is fully faithful (Haus is a full subcategory) but does not preserve epimorphisms (and hence $F^{\mathrm{op}} : \mathsf{Haus} \hookrightarrow \mathsf{Top}$ does not preserve monomorphisms). In the category Haus, epimorphisms are continuous functions with dense images. If $f : x \to y \in \mathsf{Haus}$ is an epimorphism, that means that whenever $gf = hf$, then $g = h$. This implies that $f(x)$ is dense in y. However, in Top, epimorphisms are surjective continuous maps. Maps with dense image are not the same as surjective maps, so faithful functors do not necessarily preserve epimorphisms or monomorphisms. □