Long exact sequence of homotopy groups for a fibration

Presentation notes

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Presentation notes guideline:

- To be written on the board = this typeface.
- To be said out loud = this typeface.

1 Homotopy groups: Definitions and basic constructions

- $I^n \subseteq \mathbb{R}^n$

Def: ∂I^n: subspace of I^n of elements with at least one coordinate equal to 0 or 1.

- The elements of the boundary are called the *faces* of the unit cube.

Def: If $x_0 \in X$, the n^{th} homotopy group is:

$$\pi_n(X, x_0) = \{[f] \mid f : (I^n, \partial I^n) \to (X, x_0)\}$$

- $n = 0 : I^0 = *,\ \partial I^0 = \varnothing$

$$\Rightarrow \quad \pi_0(X, x_0) = \{\text{set of path components of } X\}$$

- If $n = 1 : I^1 = [0, 1], \partial I^1 = \{0, 1\} \Rightarrow \pi_1(X, x_0)$ is the fundamental group.

1.0.1 Sum operation in $\pi_n(X, x_0)$, $n \geq 2$.

- Generalizes the operation in π_1:

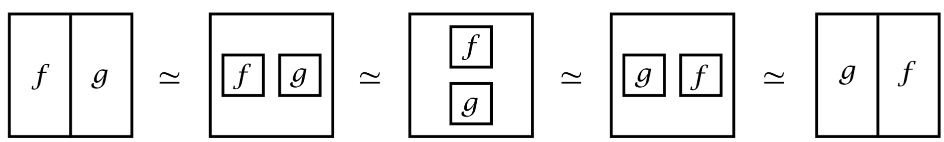
$$(f+g)(s_1, s_2, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n), & s_1 \in \left[0, \frac{1}{2}\right] \\ g(2s_1 - 1, s_2, \ldots, s_n), & s_1 \in \left[\frac{1}{2}, 1\right] \end{cases}$$

  - This is well-defined on homotopy classes.

- Since only the first coordinate is involved in the sum operation, the same arguments as for π_1 show that $\pi_n(X, x_0)$ is a group.
  - Identity: the constant map $I^n \mapsto x_0$
  - Inverses: $-f(s_1, s_2, \dots, s_n) = f(1 - s_1, s_2, \dots, s_n)$.
- $\pi_n(X, x_0)$ is *abelian* for all $n \geq 2$.

  By the definition of the operation, it suffices to show commutativity of the operation by restricting focus to just the first two coordinates s_1 and s_2, and fixing all others.

$$f + g \simeq g + f \qquad \text{via:}$$



(1) Shrink the domains of f and g to smaller subcubes (where the region outside the subcubes are mapped to the basepoint).

(2) Slide the subcubes around anywhere in I^n as long as they stay disjoint (can be done for $n \geq 2$ since both coordinates can be shifted).

(3) Enlarge the domains to original size.

1.1 Relative homotopy groups

- Let $A \subseteq X$, $x_0 \in A$.
- Denote

$$I^{n-1} = \{(x_1, x_2, \dots, x_{n-1}, 0) \mid x_i \in I\}$$

- Define

$$J^{n-1} := \overline{(\partial I^n - I^{n-1})}$$

i.e., J^{n-1} is the closure of the union of the remaining faces of I^n.

Def: n^{th} relative homotopy group of (X, A) at basepoint x_0 is

$$\pi_n(X, A, x_0) = \{[f] \mid f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)\}$$

where $f : I^n \to X$ is such that:

$$f(\partial I^n) \subseteq A,$$
$$f(J^{n-1}) = \{x_0\}$$

Remark: $\pi_n(X, x_0, x_0) = \pi_n(X, x_0)$

– i.e, Absolute homotopy groups are a special case of relative homotopy groups (analogous to absolute vs relative homology groups).

1.1.1 Sum operation in $\pi_n(X, A, x_0)$, $n \geq 2$.

- Since s_n is "no longer available", the sum operation in $\pi_n(X, A, x_0)$ is not abelian for $n = 2$ (the operation allows "movement" in every direction except the direction of s_n).
- $\pi_1(X, A, x_0)$ is not a group.
  - $I^1 = [0, 1]$, $I^0 = \{0\}$, $J^0 = \{1\}$.
    $\Rightarrow \pi_1(X, A, x_0) =$ set of homotopy classes of paths in X from a varying point in A to a fixed basepoint $x_0 \in A$
  - In general, this is not a group in any natural way.
- $\pi_n(X, A, x_0)$ is an abelian group for all $n \geq 3$.

2 Long exact sequence of relative homotopy groups

The most useful feature of relative homotopy groups is the fact that they form an exact sequence of groups.

Theorem 4.3. *The sequence given by*

$$\cdots \longrightarrow \pi_n(A, x_0) \xrightarrow{\iota_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0)$$

$$\xrightarrow{\delta} \pi_{n-1}(A, x_0) \xrightarrow{\iota_*} \pi_{n-1}(X, x_0) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_0(X, x_0)$$

is an exact sequence.

$\iota : (A, x_0) \hookrightarrow (X, x_0)$, $j : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$ are inclusion maps and $\iota_*([f]) = [\iota f]$, $j_*([f]) = [jf]$.

$\delta : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ is the boundary map that takes $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ and restricts it to I^{n-1}, i.e.,

$$\delta([f]) = [f|_{I^{n-1}}]$$

Proof.

Before we begin to show some of the six inclusions, the following will be a useful criterion to identify trivial elements of the relative homotopy group.

Compression Criterion:

A map $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ is 0 in $\pi_n(X, A, x_0)$ if and only if $f \simeq g$ relative to ∂I^n, where g has image contained in A.

– $f \simeq g$ relative to ∂I^n: there exists a homotopy f_t such that $f \simeq g$, where $f_t|_{\partial I^n}$ is independent of t

– This is analogous to the defintion of a deformation retract, where a homotopy $f_t : X \to Y$ is such that $f_t|_A$ is the identity map.

– Showing this criterion is easier done by looking at maps from D^n rather than the unit cube.

(1) $\mathrm{im}\iota_* \subseteq \ker j_*$.

$j_*\iota_*([f]) = 0$ for every $f \in \pi_n(A, x_0)$ since if

$$f : (I^n, \partial I^n, J^{n-1}) \to (A, x_0)$$

$\Rightarrow \quad f = 0$ in $\pi_n(X, A, x_0)$ by the compression cirterion. i.e., $f \simeq f$ and $f(I^n) \subseteq A$.

(2) $\ker j_* \subseteq \mathrm{im}\iota_*$.

By the compression criterion, $[f] = 0$ in $\pi_n(X, A, x_0)$ implies that $f \simeq g$ *relative to* ∂I^n with $g(I^n) \subseteq A$.

$$\Rightarrow \quad \iota_*([g]) = [f]$$

Hence, $[f] \in \mathrm{im}\iota_* \Rightarrow \ker j_* \subseteq \mathrm{im}\iota_*$.

$\therefore \ker j_* = \mathrm{im}\iota_*$.

(3) $\mathrm{im} j_* \subseteq \ker \delta$.

If $[f] \in \pi_n(X, A, x_0) \in \mathrm{im} j_*$, then

$$f|_{I^{n-1}} \text{ is constant since } f|_{I^{n-1}} = x_0$$

$$\Rightarrow \quad \delta([f]) = \text{ constant}$$

$\Rightarrow \quad \mathrm{im} j_* \subseteq \ker \delta$.

(4) $\ker \delta \subseteq \mathrm{im} j_*$.

If $\delta([f]) = 0$ where $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$,

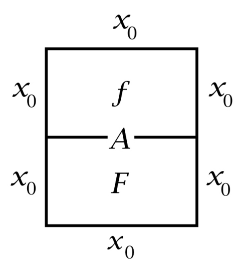
$$\Rightarrow \quad f|_{I^{n-1}} \simeq \text{ constant} \in \pi_{n-1}(A, x_0)$$

Then, $f|_{I^{n-1}} \simeq g$ with $g(I^{n-1}) = \{x_0\}$ via a homotopy

$$F : I^{n-1} \times I \to A$$

We can tack F onto f to get a map

$$h : (I^n, \partial I^n, J^{n-1}) \to (X, x_0)$$



Considering h as a map $h : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, x_0)$,

$$h \simeq f$$

via homotopy that tacks on increasingly longer segments of F.

$$\Rightarrow \quad [f] = j_*([h])$$

$\Rightarrow \quad \ker \delta \subseteq \text{im} j_*.$

$\therefore \text{im} j_* = \ker \delta$

$\square$

3 Fibrations

- The motivation behind fiber bundles is that it is a type of map that gives a long exact sequence of homotopy groups.
  - Its distinguishing feature is that all of its fibers are homeomorphic.
  - They arise in all sorts of geometry such as tangent bundles.
  - They are a generalization of covering spaces where fibers do not have to be discrete, but they don't have the unique lifting property.

<u>Homotopy lifting property wrt X</u>: A map $p : E \to B$ has this property iff $\forall$ homotopies $g_t : X \to B$ and $\tilde{g}_0 : X \to E$ lifting g_0, then $\exists$ a homotopy $\tilde{g}_t : X \to E$ lifting g_t:

$$\begin{array}{ccc} & & E \\ & \nearrow & \downarrow p \\ X & \longrightarrow & B \end{array}$$

$p : E \to B$ is called a <u>fibration</u> if it has this property wrt all spaces X.

- Every fiber bundle is a fibration and every covering space is a fibration

Theorem 4.41. *Suppose* $p : E \to B$ is a fibration. *If* $b_0 \in B, x_0 \in F = p^{-1}(b_0)$. Then,

$$p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0) \textit{ is an isomorphism}$$

Hence, if B is path-connected, $\exists$ a LES:

$$\cdots \longrightarrow \pi_n(F, x_0) \xrightarrow{\iota_*} \pi_n(E, x_0) \xrightarrow{j_*} \pi_n(B, b_0) \xrightarrow{\delta} \pi_{n-1}(F, x_0) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_0(E, x_0)$$

Example: Hopf Fibration

Let $E = S^3, B = S^2, F = S^1$.

Define $p : S^3 \to S^2$ by

$$p : (z_1, z_2) \mapsto (2z_1\overline{z_2}, |z_1|^2 - |z_2|^2)$$

where

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

and

$$S^2 = \{(z, x) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + x^2 = 1\}$$

This map p is a fiber bundle (all fibers are homeomorphic – S^1) and hence a fibration.

By Theorem 4.41, this is exact:

$$\pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) \to \pi_1(S^3)$$

<u>Fact</u>: $\pi_i(S^3) = 0$ for $i = 1, 2$ and $\pi_i(S^1) = 0$ for all $i \geq 2$.

$$\Rightarrow \quad 0 \to \pi_2(S^2) \to \mathbb{Z} \to 0 \text{ is an exact sequence}$$

$$\Rightarrow \quad \pi_2(S^2) \cong \pi_1(S^1) \cong \mathbb{Z}$$

Also, this sequence is exact:

$$\pi_3(S^1) \to \pi_3(S^3) \to \pi_3(S^2) \to \pi_2(S^1)$$

$$\Rightarrow \quad 0 \to \pi_3(S^3) \xrightarrow{\cong} \pi_3(S^2) \to 0$$

<u>Fact</u>: $\pi_3(S^3) \cong \mathbb{Z}$

$$\Rightarrow \quad \pi_3(S^2) \cong \mathbb{Z}$$

– This is not analogous to homology where higher dimensional homology groups were trivial (i.e., $H_n(S^k) = 0$ for all $n > k$).