The geometry of random graphs with a Markov flavour

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Mohabat Tarkeshian

Western University

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Overview

1. Random graphs

2. Encoding dependence

3. The stability of ERGMs

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Random graphs

What are random graphs?

a set of graphs + some (measurable) uncertainty

a set of graphs + a probability distribution

Random subgraphs of K_m

- A random subset $S \subseteq E := E(K_m)$: a random variable that assigns a probability to each subset of edges
- Equivalently, if $E = [n] = \{1, 2, \ldots, n\}$, then a random subset $S \subseteq E$ is a random element of $\mathcal{P}([n])$

Generating polynomials

- Well-developed ([BBL09], [ALGV19]) dictionary

$$\{\text{probability distributions}\} \longleftrightarrow \{\text{multivariate polynomials}\}$$

- def. For X a random subgraph, its generating polynomial is

$$F_X := \sum_{S \subseteq E} P(X = S)\mathbf{x}^S$$

where $\mathbf{x}^S = \prod_{i \in S} x_i$ and $\mathbf{x} = (x_1, \ldots, x_n)$

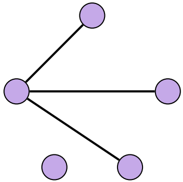
- def. A nonzero polynomial $F \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ is (real) stable if it does not have any roots in the open upper half of the complex plane $\mathcal{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$.

Guiding question

When is F_X stable?

The first ideas and strides: Erdős-Rényi

- Erdős-Rényi graphs $G(m, p)$ for $m \in \mathbb{N}, 0 < p < 1$
  1. Start with m vertices



  2. Draw an edge between each pair of vertices with independent probability p
- The giant component for $mp \to c > 1$
  - ▷ All other components will contain at most $O(\log m)$ vertices
- Limitations: modelling complex systems
  - ▷ Independence is uncommon in networks

Erdős-Rényi graphs

- For $S \subseteq E$, $P(X = S) = p^{|S|}(1-p)^{n-|S|}$

Proposition

If X is Erdős-Rényi, then $F_X = \prod_{e \in [n]}(px_e + (1-p))$ and this is a stable polynomial.

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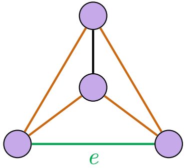
Dependence

- Each edge in the Erdős-Rényi model is independent of the rest
- Markov assumption: Adjacent edges are dependent



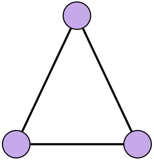
- def. A neighbourhood clique is a subset of pairwise dependent edges.

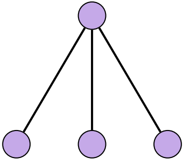
Markov random graphs

- def. The Markov random graph model on $\mathcal{P}([n])$ is:

$$P(X = G) \propto \exp\left(a t_3(G) + \sum_{k<m} b_k s_k(G) \right)$$

  - $a, b_k \in \mathbb{R}$: parameters
  - $t_3(G) := \#$ triangles in G
  - $s_k(G) := \#$ k-stars in G


triangle


3-star

- Hammersley-Clifford theorem [Bré20]

Markov random graphs

Proposition

If $n = 3$ and $a = 0$, then F_X is stable for any $b_1, b_2 \in \mathbb{R}$.

Proposition

If $n = 4$, then the stability of F_X does not depend on the edge and 2-star parameters b_1 and b_2.

Exponential random graphs (ERGMs)

- We can define a more general model with the following parameters:
  - ▷ $T > 0$: temperature parameter
  - ▷ F : a finite set of test graphs H
  - ▷ $\{a_H\}_{H \in F} \in \mathbb{R}^F$: a real parameter for each test graph
- def. $n_H(G) := \#\,\mathrm{Inj}(H, G) = \#\{H \hookrightarrow G\}$
- def. The exponential random graph model (ERGM) with the above parameters is

$$P(X = G_T) \propto \exp\left(\frac{1}{T} \sum_{H \in F} a_H n_H(G)\right)$$

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Behaviour with respect to temperature

Variational Question 1

Fix $\{a_H\}_{H \in F}$. For which T is F_{G_T} stable?

Variational Question 2

Fix T. For which parameters $\{a_H\}_{H \in F}$ is F_{G_T} stable?

Variational Question I

Theorem 1

Let X be an ERGM. If $a_H > 0$ for all $H \in F$ and $T > 0$ is sufficiently large, then F_{G_T} is stable.

- What about other parameter values and their relationship to the temperature threshold?

Variational Question 2

Theorem 2

Let X be a Markov ERGM and fix $T > 0$. Then, F_G is stable for one b_2 iff it is stable for all b_2.

Proof sketch:

- Replace subgraph counts with homomorphism counts [Lov12]
- hom$(S_k, G) = \sum_{i \in V(G)} \deg(i)^{k-1}$
- hom$(S_2, G) = 2|E(G)|$
- Specialize to a univariate polynomial [BBL09]
- Apply the Hermite-Sylvester theorem [Nat19]: a univariate polynomial is real-rooted iff its Hermite matrix is positive semidefinite

Thank you!

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