The geometry of Markov random graphs

Geometry and Combinatorics Seminar

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Overview

1. Random graphs
2. Encoding dependence
3. The stability of ERGMs
4. From stable to Lorentzian

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Random graphs

What are random graphs?

a set of graphs + some (measurable) uncertainty

a set of graphs + a probability distribution

Probability on a finite set

- Ω : a finite set
- $X : \Omega \to \{0, 1\}$: discrete random variable
- $P : \Omega \to [0, 1]$: a probability measure on Ω
  - $P(X = \Omega) = 1$

Random subgraphs

- $G = (V, E)$: a finite (undirected) graph
- $\mathcal{P}(E)$: power set of E
- A random subset $S \subseteq E$: a random variable $X : \mathcal{P}(E) \to \{0, 1\}$ that assigns a probability to each subset of edges
  - ▷ A random subset $S \subseteq E$ is a random element of $\mathcal{P}(E)$

Generating polynomials

- Well-developed ([BBL09], [ALGV19]) dictionary

$$\{\text{multivariate polynomials}\} \longleftrightarrow \{\text{probability distributions}\}$$

- def. For X a random subgraph, its generating polynomial is

$$g_X := \sum_{S \subseteq E} P(X = S)\mathbf{x}^S$$

where $\mathbf{x}^S = \prod_{i \in S} x_i$ and $\mathbf{x} = (x_e)_{e \in E}$

Enhancing the dictionary

Operations

- Multiplication $\longleftrightarrow$ disjoint union
  - $g_X \cdot g_Y = g_{X \sqcup Y}$
- Partial differentiation $\longleftrightarrow$ conditioning
  - $\partial_i g_X = \sum_{S \ni i} P(X = S)\mathbf{x}^{S \setminus \{i\}}$
  - i.e., $X \longmapsto (X \mid i \in S)$
- Specialization $\longleftrightarrow$ conditioning
  - $g_X|_{x_i=0} = \sum_{S \not\ni i} P(X = S)\mathbf{x}^S$
  - i.e., $X \longmapsto (X \mid i \notin S)$

Enhancing the dictionary

Properties

- Positive coefficients $\longleftrightarrow$ positive distribution
- Stability $\longleftrightarrow$ negative dependence
  - ▷ Modelling repelling particles
  - ▷ Pairwise negative correlation: For $i \neq j$:

$$P(i, j \in S) \leq P(i \in S) \cdot P(j \in S)$$

Stable polynomials

- def. A nonzero polynomial $g \in \mathbb{R}[\mathbf{x}]$ is (real) stable if it does not have any roots in the open upper half of the complex plane $\mathcal{H} = \{\mathbf{z} \in \mathbb{C}^E : \text{Im}(z_e) > 0 \text{ for all } e\}$.
- def. A probability distribution is strongly Rayleigh if g_X is stable.

Proposition [Brä07]

A multiaffine polynomial $g \in \mathbb{R}[\mathbf{x}]$ is stable if and only if for all $\mathbf{x} \in \mathbb{R}^E$ and $i, j \in E$ such that $i \neq j$,

$$\frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{x}) g(\mathbf{x}) \leq \frac{\partial g}{\partial x_i}(\mathbf{x}) \frac{\partial g}{\partial x_j}(\mathbf{x})$$

Lorentzian polynomials

- def. A subset $J \subseteq \mathbb{N}^E$ is M-convex when it satisfies the symmetric basis exchange property:

  For any $\alpha, \beta \in J$ and an index i such that $\alpha_i > \beta_i$, there exists an index j such that $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$ and $\beta - e_j + e_i \in J$

Examples

- $G = (V, E)$: a finite connected graph
  - $J = \{\text{spanning trees of } G\} \subseteq \{0, 1\}^E$
- $M =$ matroid on a finite ground set E
  - $J = \{\text{bases of } M\}$
- $J = \Delta_E^d \subseteq \mathbb{N}^E$
  - d^{th} discrete simplex
  - Vectors with coordinate sum d

Lorentzian polynomials

- def. The support of a polynomial $g \in \mathbb{R}[\mathbf{x}]$ is

$$\text{supp}(g) := \{S \subseteq E : c_S \neq 0\} \subseteq \mathbb{N}^E$$

  where $g(\mathbf{x}) = \sum_{S \subseteq E} c_S \mathbf{x}^S$

- def. A polynomial $g \in \mathbb{R}[\mathbf{x}]$ is called positive if $\text{supp}(g) = \{0, 1\}^E$

Lorentzian polynomials

- Notation
  - ▷ H_E^d : homogeneous polynomials of degree d in variables $(x_e)_{e \in E}$
  - ▷ M_E^d : polynomials in H_E^d whose supports are M-convex
  - ▷ $L_E^2 \subseteq H_E^2$: quadratic forms with non-negative coefficients that have at most one positive eigenvalue
- def. A homogeneous polynomial $h \in H_E^d$ is Lorentzian if its support is M-convex and $\partial_i h \in L_E^{d-1}$ for all $i \in E$. i.e., for $d > 2$:

$$L_E^d := \{h \in M_E^d : \partial_i h \in L_E^{d-1} \text{ for all } i \in E\}$$

Lorentzian probability measures

- def. The homogenization of g_X is

$$h_X(z, \mathbf{x}) := \sum_{S \subseteq E} P(X = S) z^{|E|-|S|} \mathbf{x}^S$$

- def. A probability distribution is Lorentzian if h_X is Lorentzian.

Proposition [BH20]

If X is strongly Rayleigh, then X is Lorentzian.

A word on negative dependence

Proposition [BBL09]

Strongly Rayleigh probability measures are strongly conditionally negatively associated and pairwise negatively correlated.

Proposition [BH20]

Lorentzian probability measures are 2-Rayleigh: for all $\mathbf{x} \in \mathbb{R}^E_{\geq 0}$ and $i, j \in E$ such that $i \neq j$,

$$\frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{x}) g(\mathbf{x}) \leq 2 \left(\frac{\partial g}{\partial x_i}(\mathbf{x}) \frac{\partial g}{\partial x_j}(\mathbf{x}) \right)$$

A word on negative dependence

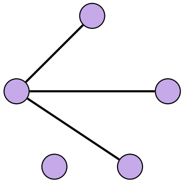
- Negatively dependent probability measures have applications to determinantal point processes and machine learning ([AGV21, KT12])
- Identifying negatively dependent measures ([Pem00])

Guiding question

When is g_X stable or Lorentzian?

The first ideas and strides: Erdős-Rényi

- $G = (V, E)$: finite
- Erdős-Rényi graphs $G(p)$ for $0 < p < 1$
    1. Start with vertices V



    2. Draw an edge between each pair of vertices with independent probability p
- Limitations: modelling complex systems

Erdős-Rényi graphs

- For $S \subseteq E$, $P(X = S) = p^{|S|}(1-p)^{|E|-|S|}$

Proposition

If X is Erdős-Rényi, then $g_X = \prod_{e \in E}(px_e + (1-p))$ and this is a stable polynomial.

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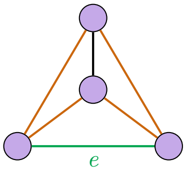
Dependence

- Each edge in the Erdős-Rényi model is independent of the rest

How do we define a joint distribution where the random variables are not independent?

- Markov assumption: Adjacent edges are dependent



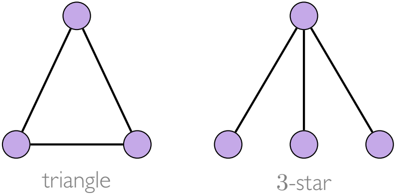
- def. A neighbourhood clique is a subset of pairwise dependent edges.

Markov random graphs

- $G = (V, E)$: finite (undirected) graph with no self-loops
- def. The Markov random graph model on $\mathcal{P}(E)$ is:

$$P(X = G_S) \propto \exp\left(a t_3(G_S) + \sum_{k \geq 1} b_k s_k(G_S) \right)$$

- ▷ $a, b_k \in \mathbb{R}$: parameters
- ▷ $t_3(G_S) := \#$ triangles in G_S
- ▷ $s_k(G_S) := \#$ k-stars in G_S



- def. The reduced Markov random graph is the above model reduced to only considering the edge and 2-star parameters.

The Markov-Gibbs correspondence

{positive Markov random fields} $\longleftrightarrow$ {finite Gibbs distributions}

Hammersley-Clifford theorem (1971)

A collection of positive random variables satisfy a Markov property if and only if it is a (finite) Gibbs distribution:

$$P(X = S) \propto \exp\left(-\mathcal{E}(S)\right)$$

where $\mathcal{E}$ is an energy function that encodes the neighbourhood dependencies

- Every Gibbs distribution is positive

Cubic stable Markov random graphs

Proposition

If $G = C_3$, then the Markov random graph model is stable if and only if the triangle parameter $a = 0$.

Exponential random graphs (ERGMs)

- We can define a more general model with the following parameters:
  - $T > 0$: temperature parameter
  - F : a finite set of test graphs H
  - $\{a_H\}_{H \in F} \in \mathbb{R}^F$: a real parameter for each test graph
- def. $n_H(G_S) := \# \,\mathrm{Inj}(H, G_S) = \#\{H \hookrightarrow G_S\}$
- def. The exponential random graph model (ERGM) with the above parameters is

$$P(X = G_S) \propto \exp\left(\frac{1}{T} \sum_{H \in F} a_H n_H(G_S)\right)$$

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Behaviour with respect to temperature

- X : an ERGM

Variational question 1

Fix $\{a_H\}_{H \in F}$. For which T is g_X stable?

Variational question 2

Fix T. For which parameters $\{a_H\}_{H \in F}$ is g_X stable?

Variational question 1

Theorem 1

Let X be an ERGM. If $a_H > 0$ for all $H \in F$, then $\lim_{T\to\infty} g_X$ is stable.

Variational question 2

Theorem 2

Let X be a reduced Markov ERGM and fix $T > 0$. Then, X is strongly Rayleigh for any choice of parameters b_1 and b_2.

Proof sketch:

- Replace subgraph counts with homomorphism densities [Lov12]
- $\hom(S_k, G) = \sum_{i \in V(G)} \deg(i)^{k-1}$
- $\hom(S_2, G) = 2|E(G)|$
- Specialize to a univariate polynomial [BBL09]
- Apply the Hermite-Sylvester theorem [Nat19]: a univariate polynomial is real-rooted iff its Hermite matrix is positive semidefinite

□

ERGMs on cycle graphs

Corollary

If $n > 3$, $G = C_n$, and X is the Markov random graph model on G, then X is strongly Rayleigh for any choice of parameters b_1 and b_2.

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Lorentzian distributions and negative dependence

- Lorentzian is weaker than stability (but easier to test)
- Some negative dependence (i.e., the 2-Rayleigh property)

Proposition

If X is positive, then the support of h_X is M-convex.

Corollary

If X is an ERGM, then the support of h_X is M-convex.

Testing the Lorentzian condition for ERGMs

- Determining whether an ERGM is Lorentzian amounts to determining the signature of quadratic forms
  - Quadratic forms $\longleftrightarrow$ square symmetric matrices
- Q : $n \times n$ symmetric matrix
- $(n_+(Q), n_-(Q), n_0(Q))$: the number of positive, negative, and zero eigenvalues of Q

Testing the Lorentzian condition for ERGMs

- Q : $n \times n$ symmetric matrix
- $D_k(Q)$: leading $k \times k$ principal minor of Q

Proposition

If $D_k(Q) \neq 0$ for all k and $D_1(Q) > 0$, then

$$\begin{aligned} n_0(Q) &= 0 \\ n_+(Q) &= \left| \left\{ 1 \leq k \leq n : \frac{D_k(Q)}{D_{k-1}(Q)} > 0 \right\} \right| \\ n_-(Q) &= \left| \left\{ 1 \leq k \leq n : \frac{D_k(Q)}{D_{k-1}(Q)} < 0 \right\} \right| \end{aligned}$$

where $D_0(Q) := 1$.

Testing the Lorentzian condition for ERGMs

- A quadratic form satisfies the Lorentzian condition iff $n_0(Q) = 0$ and $n_+(Q) \leq 1$

Corollary

If X is an ERGM and Q is a (relevant) quadratic form, then it satisfies the Lorentzian condition iff $D_k(Q) > 0$ for all odd k and $D_k(Q) < 0$ for all even k.

Lorentzian cubic Markov random graphs

Proposition

If $G = C_3$ then the Markov random graph model is Lorentzian if and only if the triangle parameter $a < \frac{9T}{2}\ln 2$.

Proof sketch:

- For ease of notation, let $B_1 := \exp\left(\frac{b_1}{3T}\right)$, $B_2 := \exp\left(\frac{b_2}{9T}\right)$, and $A := \exp\left(\frac{a}{27T}\right)$

$$h_X = z^3 + B_1^2 B_2^2 z^2 \left(\sum_{i=1}^{3} x_i\right) + B_1^4 B_2^4 z \left(\sum_{\{i,j\}\in[3]} x_i x_j\right) + B_1^6 B_2^6 A^6 x_1 x_2 x_3$$

Lorentzian cubic Markov random graphs: proof

- If $Q_0 = \frac{\partial}{\partial z} h_X$, then

$$D_1(Q_0) = 3$$
$$D_2(Q_0) = -4B_1^4 B_2^4 < 0$$
$$D_3(Q_0) = 5B_1^8 B_2^8 > 0$$
$$D_4(Q_0) = -6B_1^{12} B_2^{12} < 0$$

- The first relevant quadratic form satisfies the coniditon for any choice of edge and 2-star parameters b_1 and b_2

Lorentzian cubic Markov random graphs: proof

- For $i \in [3]$, let $Q_i = \frac{\partial}{\partial x_i} h_X$

$$D_1(Q_i) = B_1^2 B_2^2 > 0$$
$$D_2(Q_i) = -B_1^8 B_2^8 < 0$$
$$D_3(Q_i) = B_1^{14} B_2^{14} A^6 \left(2 - A^6\right)$$

- The remaining quadratic forms Q_i satisfy the Lorentzian condition iff

$$2 - A^6 = 2 - \exp\left(\frac{2a}{9T}\right) > 0$$
$$a < \frac{9T}{2} \ln 2$$

□

Future directions

- Sequences of graphs as $V(G) \to \infty$
- Alternating-sign models
- ERGMs on finite graphs: acyclic condition
    - The random cluster model and the Tutte polynomial
- Connections to spin glass models and random matrix theory

Thank you!

References I

N. Anari, S. O. Gharan, and C. Vinzant.
Log-concave polynomials i: Entropy and a deterministic approximation algorithm for counting bases of matroids.
Duke Mathematical Journal, October 2021.

N. Anari, K. Liu, S. O. Gharan, and C. Vinzant.
Log-concave polynomials ii: High-dimensional walks and an fpras for counting bases of a matroid.
STOC 2019: Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, June 2019.

References II

J. Borcea, P. Brändén, and T.M. Liggett.
Negative dependence and the geometry of polynomials.
Journal of the American Mathematical Society, 22(2):521–567, April 2009.

P. Brändén and J. Huh.
Lorentzian polynomials.
Annals of Mathematics, 192(3):821–891, November 2020.

P. Brändén.
Polynomials with the half-plane property and matroid theory.
Advances in Mathematics, 216:302–320, June 2007.

References III

A. Kulesza and B. Taskar.
Determinantal point processes for machine learning.
Foundations and Trends in Machine Learning, 5(2-3):123–286, December 2012.

L. Lovász.
Large networks and graph limits.
American Mathematical Society Colloquium Publications, December 2012.

M. B. Nathanson.
The hermite-sylvester crtierion for real-rooted polynomials.
arXiv:1911.01745v2 [math.CO], November 2019.

References IV

R. Pemantle.

Towards a theory of negative dependence.

Journal of Mathematical Physics, 41(1371), March 2000.